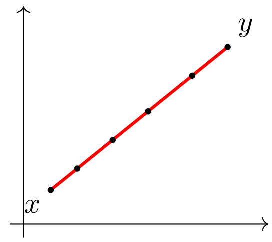


Convex: every chord stays inside.

Figure 1: Geometric intuition of convexity.

Week 3 — Convexity

1. Basic definitions

Definition (Convex set)**.** Let $X \subseteq \mathbb{R}^J$ be nonempty. We call X *convex* if for every $x, y \in X$ and every $\alpha \in [0,1]$,

$$\alpha x + (1-\alpha)y \in X.$$

The point $\alpha x + (1-\alpha)y$ is the *line segment* point between x and y.

Definition (Convex combination)**.** Given points $x^1, \dots, x^k \in \mathbb{R}^J$, a vector

$$z = \sum_{i=1}^{k} \alpha_i x^i$$

is a *convex combination* of the x^i if $\alpha_i \geq 0$ for all i and $\sum_{i=1}^{k} \alpha_i = 1$.

Definition (Convex hull)**.** For $S \subseteq \mathbb{R}^J$, the *convex hull* $\operatorname{conv}(S)$ is the set of all finite convex combinations of points in S.

Remark. Geometrically, $\operatorname{conv}(S)$ is "everything in S plus exactly the extra line-segment pieces needed to make it convex."

2. Convex subsets of $\mathbb{R}$ are exactly intervals

Theorem 1. *A nonempty set $X \subseteq \mathbb{R}$ is convex if and only if it is an interval.*

What we need to show.

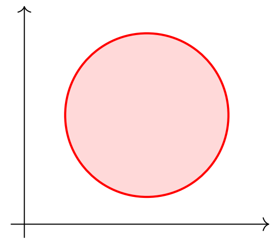
- ($\Rightarrow$) If X is convex, then whenever $a < b$ are in X, the whole $[a, b] \subseteq X$.
- ($\Leftarrow$) If X is an interval, then for any $x, y \in X$ and $\alpha \in [0,1]$, the point $\alpha x + (1-\alpha)y$ is still in X.

Proof. ($\Rightarrow$) Take $a < b$ in X. For any $t \in [0,1]$, convexity gives $ta + (1-t)b \in X$. As t runs from 0 to 1 this sweeps $[b, a] = [a, b]$ (reversing orientation), hence $[a, b] \subseteq X$.

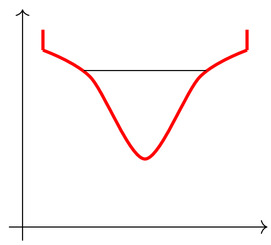
($\Leftarrow$) Suppose X is an interval and take $x, y \in X$. Without loss of generality $x \leq y$. For $\alpha \in [0,1]$,

$$\alpha x + (1-\alpha)y \in [x, y] \subseteq X,$$

since intervals contain every point between their endpoints. Thus X is convex. $\square$



Disk: convex



Chord exits the set $\Rightarrow$ not convex

Figure 2: Convex vs. non-convex geometry.

Remark (Contrapositive method you saw in class)**.** If $X \subseteq \mathbb{R}$ is *not* an interval, there exist $x < \bar{x}$ in X and a $z \in (x, \bar{x})$ with $z \notin X$. Define $f(\alpha) = \alpha x + (1-\alpha)\bar{x}$, a continuous map from $[0,1]$ onto $[x, \bar{x}]$. The Intermediate Value Theorem yields $\alpha^* \in (0,1)$ with $f(\alpha^*) = z$. If X were convex, z would have to be in X; contradiction.

3. Canonical examples and non-examples

- **Convex:** any line segment; any Euclidean ball (disk) $\{x : \|x - x_0\|_2 \leq r\}$; any halfspace $\{x : a^\top x \leq b\}$; the epigraph of a convex function $\{(x,t) : t \geq f(x)\}$.
- **Non-convex:** an annulus; a "crescent" shape; a set with a "hole"; the hypograph of a convex (i.e., not concave) function.

4. Structural properties of convex sets

Proposition (Intersection)**.** *The intersection of any family of convex sets is convex. In particular, $X_1 \cap X_2$ is convex (or empty).*

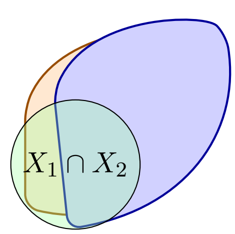


Figure 3: Intersection stays convex.

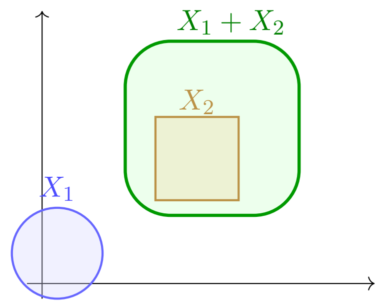


Figure 4: Minkowski sum: $X_1 + X_2 = (c + r\mathbb{B}_2) + X_2$. (Rounded corners have radius r and the whole shape is translated by c.)

For x, y in the intersection and for $\alpha \in [0, 1]$, the convex combination lies in *each* set, hence in their intersection.

Proof. Let $\{C_\lambda\}_{\lambda\in\Lambda}$ be convex. Fix $x, y \in \cap_\lambda C_\lambda$ and $\alpha \in [0, 1]$. For each λ, convexity of C_λ gives $\alpha x + (1-\alpha)y \in C_\lambda$. Therefore $\alpha x + (1-\alpha)y \in \cap_\lambda C_\lambda$. □

Proposition (Linear images)**.** *If X is convex and A is a linear map, then AX is convex. If b is a vector, then $AX + b$ is convex as well (affine image).*

Proof. For $x, y \in X$,

$$\alpha(Ax + b) + (1-\alpha)(Ay + b) = A\big(\alpha x + (1-\alpha)y\big) + b \in AX + b.$$

□

Proposition (Scalar multiples and sums)**.** *If X is convex and $\lambda \in \mathbb{R}$, then $\lambda X = \{\lambda x : x \in X\}$ is convex. If X_1, X_2 are convex, then the Minkowski sum*

$$X_1 + X_2 = \{x_1 + x_2 : x_1 \in X_1,\ x_2 \in X_2\}$$

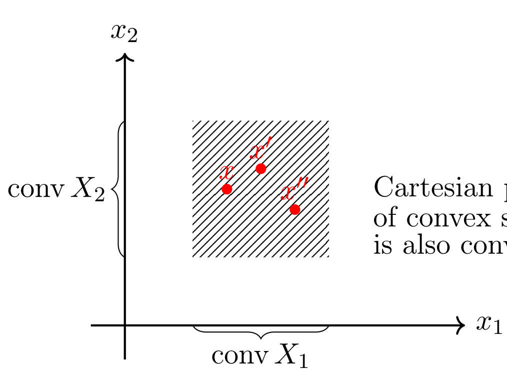
is convex. More generally, for any scalars $\lambda_1, \lambda_2 \in \mathbb{R}$, the set $\lambda_1 X_1 + \lambda_2 X_2$ is convex.

Proof idea before writing it. Use linearity and convexity in each component, then regroup.

Proof. Let $z_i = \lambda_1 x_i + \lambda_2 y_i$ with $x_i \in X_1$, $y_i \in X_2$ for $i = 1, 2$. For $\alpha \in [0, 1]$,

$$\alpha z_1 + (1-\alpha)z_2 = \lambda_1\big(\alpha x_1 + (1-\alpha)x_2\big) \ + \ \lambda_2\big(\alpha y_1 + (1-\alpha)y_2\big).$$

Convexity of X_1 and X_2 gives $\alpha x_1 + (1-\alpha)x_2 \in X_1$ and $\alpha y_1 + (1-\alpha)y_2 \in X_2$. Hence the sum lies in $\lambda_1 X_1 + \lambda_2 X_2$. □



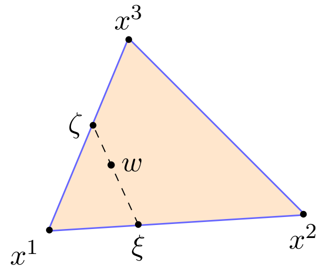


Figure 5: Two-stage convex mixing inside a triangle.

Proposition (Cartesian products)**.** *If $X_1 \subseteq \mathbb{R}^{n_1}$ and $X_2 \subseteq \mathbb{R}^{n_2}$ are convex, then $X_1 \times X_2 \subseteq \mathbb{R}^{n_1+n_2}$ is convex.*

Proof outline. Take $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$ and $\alpha \in [0, 1]$. Then

$$\alpha(x_1, y_1) + (1-\alpha)(x_2, y_2) = \big(\alpha x_1 + (1-\alpha)x_2,\ \alpha y_1 + (1-\alpha)y_2\big) \in X_1 \times X_2,$$

by convexity of each coordinate set.

5. Working with convex combinations (the triangle picture)

Proposition (Two-stage mixing produces arbitrary barycentric weights)**.** *Fix three points $x^1, x^2, x^3 \in \mathbb{R}^J$. Every point in $\operatorname{conv}\{x^1, x^2, x^3\}$ can be written as*

$$w = \alpha\,\xi + (1-\alpha)\,\zeta, \qquad \xi = \phi\,x^1 + (1-\phi)\,x^2, \quad \zeta = \psi\,x^1 + (1-\psi)\,x^3,$$

for some $\alpha, \phi, \psi \in [0, 1]$. Conversely, any such choice yields a convex combination $w = \sum_{i=1}^{3} \lambda_i x^i$ with $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$.

Before the algebra. The picture: first choose a point ξ on the edge $[x^1, x^2]$, then a point ζ on the edge $[x^1, x^3]$; and finally mix ξ and ζ. This fills exactly the triangle.

Proof. Expand:

$$\begin{aligned} w &= \alpha\big(\phi x^1 + (1-\phi)x^2\big) + (1-\alpha)\big(\psi x^1 + (1-\psi)x^3\big) \\ &= \underbrace{\big(\alpha\phi + (1-\alpha)\psi\big)}_{\lambda_1} x^1 + \underbrace{\alpha(1-\phi)}_{\lambda_2} x^2 + \underbrace{(1-\alpha)(1-\psi)}_{\lambda_3} x^3. \end{aligned}$$

Each coefficient is nonnegative and $\lambda_1 + \lambda_2 + \lambda_3 = \alpha\phi + (1-\alpha)\psi + \alpha(1-\phi) + (1-\alpha)(1-\psi) = 1$. Thus w is a convex combination. The converse (recovering (α, ϕ, ψ) from $\lambda_1, \lambda_2, \lambda_3 \geq 0$, sum 1) can be done, e.g., by taking

$$\alpha = \lambda_2 + \lambda_3, \qquad \phi = \frac{\lambda_1}{\lambda_1 + \lambda_2}, \qquad \psi = \frac{\lambda_1}{\lambda_1 + \lambda_3}$$

when $\lambda_1 + \lambda_2 > 0$, and a symmetric choice when it is 0. □

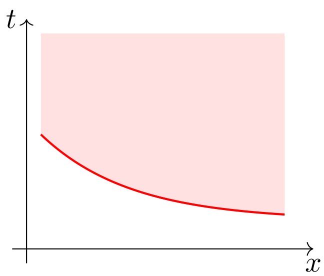


Figure 6: Epigraph of a convex f: convex set

6. Characterizations of the convex hull

Proposition. *For any $S \subseteq \mathbb{R}^J$,*

$$\operatorname{conv}(S) = \bigcap \{ C \subseteq \mathbb{R}^J : C \text{ is convex and } S \subseteq C \}.$$

Moreover, conv(S) *is convex, contains S, and is the smallest convex set that contains S.*

What we need to show.

1. Every convex set C containing S contains all finite convex combinations of S (closed under chords), hence it contains conv(S).
2. The intersection of convex sets is convex (Prop.).

Proof. Immediate from Proposition and the fact that each convex $C \supseteq S$ contains every chord between points of S, then every chord between points already added, etc. That recursive closure is exactly "finite convex combinations." □

7. Quick gallery of standard closure properties (one-liners)

- If X is convex and $a \in \mathbb{R}^J$, then $X + a = \{x + a : x \in X\}$ is convex (translation).
- If X is convex and A is linear, then $A^{-1}(X) = \{z : Az \in X\}$ is convex (inverse image under linear maps).
- If f is convex, then epi$(f) = \{(x,t) : t \geq f(x)\}$ is convex. If f is concave, hypo$(f) = \{(x,t) : t \leq f(x)\}$ is convex.

8. (Optional) nice compact proofs to keep handy

Lemma (Chord test)**.** *A set X is convex if and only if it contains the entire chord between every two of its points.*

Proof. This is the definition unpacked: $[x,y] = \{\alpha x + (1-\alpha)y : \alpha \in [0,1]\} \subseteq X$ for all $x, y \in X$. □

Lemma (Products are convex again)**.** *Already proved in Proposition ; keep it nearby: it is used constantly when moving between goods and prices, or states and controls.*

Convex and concave functions

Standing assumptions and notation

Let $X \subseteq \mathbb{R}^n$ be nonempty and convex. Let $f : X \to \mathbb{R}$.

- The *graph* of f is $\text{graph}(f) := \{(x, y) \in X \times \mathbb{R} :\ f(x) = y\}$.
- For $x_1, x_2 \in X$ and $\alpha \in [0, 1]$, write

$$x_\alpha := \alpha x_1 + (1 - \alpha)x_2, \qquad c_\alpha := \alpha f(x_1) + (1 - \alpha)f(x_2)$$

  for the point on the line segment in the domain (x_α) and the corresponding value on the chord (c_α).

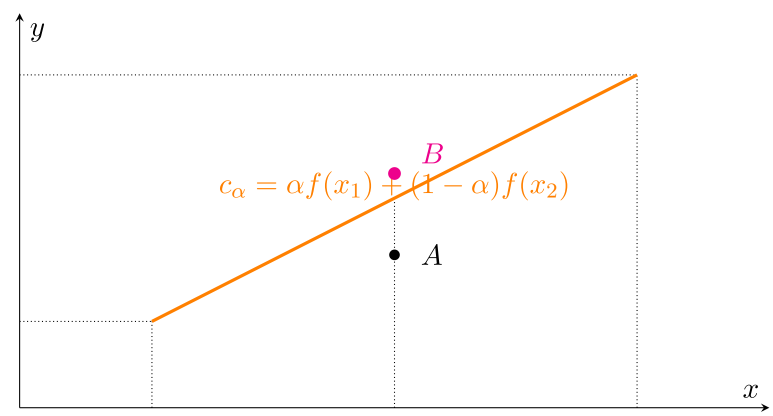


Figure 7: Given $x_1, x_2 \in X$ and $\alpha \in [0, 1]$, the point $(x_\alpha, f(x_\alpha))$ may lie *below* the chord (point A) or *above* the chord (point B). Convexity/concavity will rule out one of these possibilities.

Definition. Let $X \subseteq \mathbb{R}^n$ be nonempty and convex. Let $f : X \to \mathbb{R}$. Then:

1. f is *convex* if for each $x, y \in X$ and each $\alpha \in [0, 1]$,

$$f\big(\alpha x + (1 - \alpha)y\big) \ \le\ \alpha f(x) + (1 - \alpha)f(y).$$

2. f is *concave* if for each $x, y \in X$ and each $\alpha \in [0, 1]$,

$$f\big(\alpha x + (1 - \alpha)y\big) \ \ge\ \alpha f(x) + (1 - \alpha)f(y).$$

3. f is *strictly convex* if for each distinct $x, y \in X$ and each $\alpha \in (0, 1)$,

$$f\big(\alpha x + (1 - \alpha)y\big) \ <\ \alpha f(x) + (1 - \alpha)f(y).$$

4. f is *strictly concave* if for each distinct $x, y \in X$ and each $\alpha \in (0, 1)$,

$$f\big(\alpha x + (1 - \alpha)y\big) \ >\ \alpha f(x) + (1 - \alpha)f(y).$$

Remark. Any (affine) linear function $x \mapsto a + b \cdot x$ is both convex and concave, since the inequality above holds with equality for all x, y and α.

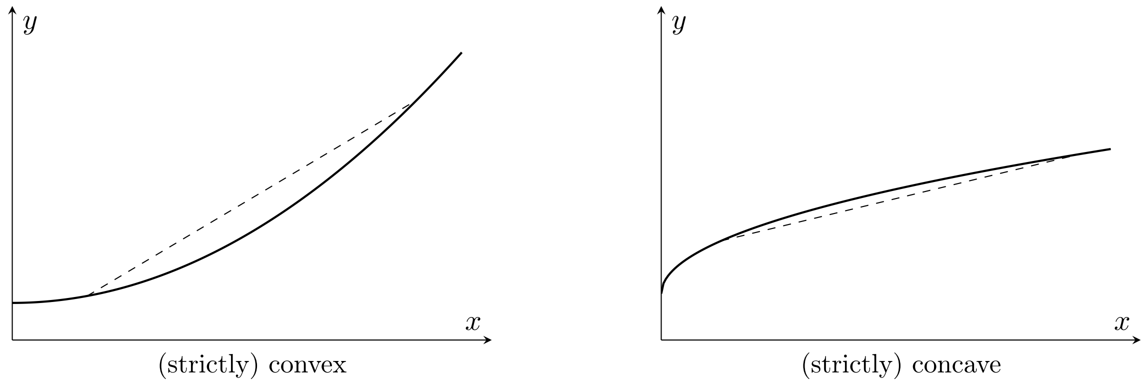


Figure 8: Convex versus concave: the function lies below (resp. above) every chord.

One-dimensional criteria

Epigraph and hypograph

We first fix a one-dimensional setting: let $X \subseteq \mathbb{R}$ be a nonempty interval and $f : X \to \mathbb{R}$. Define the *epigraph* and *hypograph* as subsets of $\mathbb{R}^2$:

$$\operatorname{epi}(f) := \{(x, y) \in X \times \mathbb{R} : \ y \geq f(x)\}, \qquad \operatorname{hypo}(f) := \{(x, y) \in X \times \mathbb{R} : \ y \leq f(x)\}.$$

Proposition (Epigraph/hypograph criterion)**.** *Let $X \subseteq \mathbb{R}$ be an interval and $f : X \to \mathbb{R}$. Then f is convex on X* iff $\operatorname{epi}(f) \subset \mathbb{R}^2$ *is a convex set. Dually, f is concave on X* iff $\operatorname{hypo}(f)$ *is convex.*

Intuition. Points of the epigraph are simply points (x, y) *above the graph.* Take two such points (x_1, y_1) and (x_2, y_2) with $y_i \geq f(x_i)$. For $\alpha \in [0, 1]$, the convex combination

$$(x_\alpha, y_\alpha) := \big(\alpha x_1 + (1 - \alpha)x_2, \ \alpha y_1 + (1 - \alpha)y_2\big)$$

belongs to $\operatorname{epi}(f)$ precisely when $y_\alpha \geq f(x_\alpha)$. Using $y_\alpha \geq \alpha f(x_1) + (1 - \alpha)f(x_2)$, this reduces to the convexity inequality

$$f(x_\alpha) \ \leq \ \alpha f(x_1) + (1 - \alpha)f(x_2).$$

For concavity, reverse the inequalities and use the hypograph.

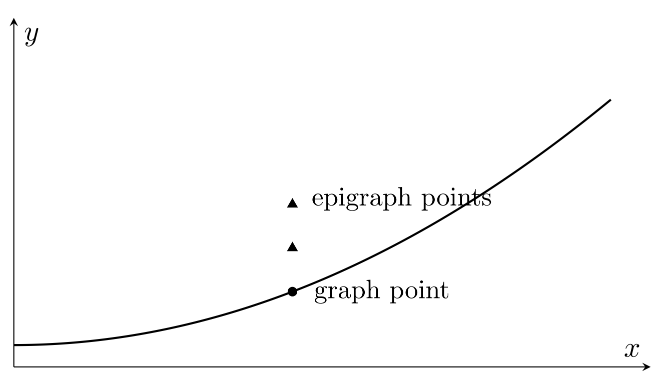


Figure 9: In $\mathbb{R} \to \mathbb{R}$, $\operatorname{epi}(f)$ consists of the points lying above the graph.

Proposition. *One-dimensional derivative criteria Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable.*

- f *is* concave *if and only if* f' *is* nonincreasing.
- f *is* convex *if and only if* f' *is* nondecreasing.
- f *is* ***strictly concave*** *if and only if* f' *is* ***strictly decreasing***.
- f *is* ***strictly convex*** *if and only if* f' *is* ***strictly increasing***.

Moreover, if f is twice differentiable, then

$$f \text{ is concave (convex)} \iff f''(x) \leq 0 \ (\geq 0) \text{ for all } x.$$

If, in addition, $f''(x) < 0$ (resp. > 0) for all x, then f is strictly concave *(resp.* strictly convex*). The converses need not hold: strict concavity/convexity allows f'' to vanish at isolated points (e.g., $f(x) = -x^4$ at $x = 0$).*

Example. (One-dimensional case and derivatives) Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -x^2$. Here $A = [-1]$ is negative definite, so f is *strictly concave*. Moreover,

$$f'(x) = -2x \quad \text{is nonincreasing in } x, \qquad f''(x) = -2 < 0.$$

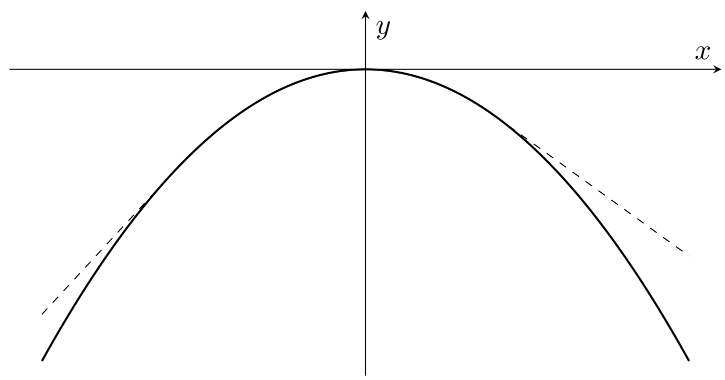


Figure 10: Example $f(x) = -x^2$: concave, with decreasing slopes (tangent lines).

Multi-dimensional criteria

Proposition. *Let $X \subseteq \mathbb{R}^n$ be nonempty and convex, and let $f : X \to \mathbb{R}$.*

1. *f is convex if and only if its epigraph*

   $$\operatorname{epi}(f) := \{(x, y) \in X \times \mathbb{R} : \ y \geq f(x)\}$$

   is a convex set.

2. *f is concave if and only if its hypograph*

   $$\operatorname{hypo}(f) := \{(x, y) \in X \times \mathbb{R} : \ y \leq f(x)\}$$

   is a convex set.

Recall. (Quadratic forms) A *quadratic form* on $\mathbb{R}^n$ can be written as

$$f(x) \;=\; x^\top A\, x,$$

for some *symmetric* matrix $A \in \mathbb{R}^{n\times n}$.

Proposition (Convexity/concavity via definiteness)**.** *Let $f(x) = x^\top Ax$ with $A = A^\top$.*

1. *f is **strictly convex** iff A is* positive definite *(PD).*
2. *f is **strictly concave** iff A is* negative definite *(ND).*
3. *f is **convex** iff A is* positive semidefinite *(PSD).*
4. *f is **concave** iff A is* negative semidefinite *(NSD).*

Proof. Fix $x, y \in \mathbb{R}^n$ and $\alpha \in [0,1]$. Let $x_\alpha := \alpha x + (1-\alpha)y$. Then

$$\begin{aligned}
f(x_\alpha) &= (\alpha x + (1-\alpha)y)^\top A(\alpha x + (1-\alpha)y)\\
&= \alpha^2\, x^\top Ax \;+\; \alpha(1-\alpha)\, x^\top Ay \;+\; \alpha(1-\alpha)\, y^\top Ax \;+\; (1-\alpha)^2\, y^\top Ay\\
&= \alpha^2\, x^\top Ax \;+\; 2\alpha(1-\alpha)\, x^\top Ay \;+\; (1-\alpha)^2\, y^\top Ay \quad \text{(because } A \text{ is symmetric).}
\end{aligned}$$

On the other hand,

$$\alpha f(x) + (1-\alpha) f(y) \;=\; \alpha\, x^\top Ax + (1-\alpha)\, y^\top Ay.$$

Subtracting gives the fundamental identity

$$\boxed{\alpha f(x) + (1-\alpha) f(y) - f(x_\alpha) = \alpha(1-\alpha)\,(x-y)^\top A\,(x-y)}$$

which shows that the sign of the difference is governed by the definiteness of A. If $A \succeq 0$ (PSD), the right-hand side is ≥ 0 and hence $f(x_\alpha) \leq \alpha f(x) + (1-\alpha) f(y)$ (convexity); if $A \succ 0$ (PD) and $x \neq y$ with $\alpha \in (0,1)$, the inequality is strict. The concave cases follow analogously with $A \preceq 0$ or $A \prec 0$. ☐

Example. Two-dimensional functions Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x) = -(x_1^2 + x_2^2)$. Then $f(x) = x^\top Ax$ with

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2.$$

Hence $x^\top Ax = -x_1^2 - x_2^2 \leq 0$ with equality only at $x = 0$. By principal minors, A is negative definite:

$$\det\bigl(-1\bigr) = -1 < 0, \qquad \det(A) = \det(-I_2) = 1 > 0.$$

Therefore f is *strictly concave*. Its gradient is

$$\nabla f(x_1, x_2) = (-2x_1,\, -2x_2).$$

Remark. Strict concavity does *not* imply that ∇f is strictly decreasing in the *coordinatewise* order. For instance, if we hold x_1 fixed and increase x_2, only the second component of ∇f decreases while the first component remains unchanged.

Example. An indefinite quadratic form Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2 \;=\; x^\top Ax, \qquad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then $\nabla f(x) = (2x_1 + 4x_2,\ 2x_2 + 4x_1)$, which is *coordinatewise increasing*: if $x' \neq x''$ and $x' \geq x''$ componentwise, then $\nabla f(x') > \nabla f(x'')$ componentwise. Nevertheless, A is *indefinite* (e.g., $f(1, 1) = 6 > 0$ while $f(1, -1) = -2 < 0$), so f is neither convex nor concave.

Intuition. Information from first derivatives alone is insufficient to infer convexity or concavity in higher dimensions: we must incorporate *interaction effects* via the cross-partials, i.e., second derivatives.

Second derivatives and the Hessian

Assume $f : X \to \mathbb{R}$ is *twice continuously differentiable* on $X \subseteq \mathbb{R}^n$; that is, all second partial derivatives exist and are continuous on X. By Clairaut's (Schwarz's) theorem,

$$\frac{\partial^2 f}{\partial x_i\, \partial x_j} \;=\; \frac{\partial^2 f}{\partial x_j\, \partial x_i} \quad \text{for all } i, j.$$

The *Hessian* of f at $x \in X$ is the symmetric $n \times n$ matrix

$$H_f(x) := \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2}(x) & \dfrac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \dfrac{\partial^2 f}{\partial x_2^2}(x) & \cdots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1}(x) & \dfrac{\partial^2 f}{\partial x_n \partial x_2}(x) & \cdots & \dfrac{\partial^2 f}{\partial x_n^2}(x) \end{pmatrix}.$$

Proposition. *Hessian characterization. Suppose $X \subseteq \mathbb{R}^n$ is* open *and convex, and $f : X \to \mathbb{R}$ is twice continuously differentiable on X.*

1. *f is convex if and only if $H_f(x)$ is positive semidefinite for every $x \in X$.*
2. *f is strictly convex if and only if $H_f(x)$ is positive definite for every $x \in X$.*
3. *f is concave if and only if $H_f(x)$ is negative semidefinite for every $x \in X$.*
4. *f is strictly concave if and only if $H_f(x)$ is negative definite for every $x \in X$.*

Intuition. The Hessian aggregates all second-order effects, including cross-partials. Positivity (or negativity) of the quadratic form $v^\top H_f(x)v$ for all directions v guarantees that along any line through x, the second derivative of the univariate restriction is nonnegative (or nonpositive), which is exactly convexity (or concavity) along every line—and hence on X.

Remark (Why the openness assumption appears with Hessians)**.** When using the Hessian characterization, we want a full neighborhood around each x_0 to apply line restrictions $t \mapsto f(x_0 + tv)$ and second-order Taylor expansions. Boundary points may fail to have such two-sided neighborhoods even if f is differentiable along some directions, hence the condition X open (or at least $x_0 \in \text{int}(X)$).

Example. (Strict concavity does not imply a negative definite Hessian everywhere) Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -x^4$. Then

$$f'(x) = -4x^3 \quad \text{is strictly decreasing in } x,$$

so by the one–dimensional criterion f is *strictly concave.* However,

$$f''(x) = -12x^2 \ \leq 0, \qquad f''(0) = 0,$$

that is, the Hessian $H_f(x) = [-12x^2]$ is *negative semidefinite* (NSD) and not negative definite (ND) at $x = 0$. Hence strict concavity does not force $H_f(x)$ to be ND at every point.

Remark. The correct Hessian test is: convex $\Leftrightarrow H_f(x) \succeq 0$ for all x; strictly convex $\Leftrightarrow H_f(x) \succ 0$ for all x; concave $\Leftrightarrow H_f(x) \preceq 0$ for all x; strictly concave $\Leftrightarrow H_f(x) \prec 0$ for all x. Strict concavity allows $H_f(x)$ to be only NSD at isolated points.

Proposition. *(Continuity on the interior) Suppose $f : X \to \mathbb{R}$ is either convex or concave. Then for every $x_0 \in \operatorname{int}(X)$, the function f is continuous at x_0. In particular, if X is open, convexity or concavity implies continuity on X.*

Intuition (Sketch via the epigraph)**.** For convex f, the epigraph is convex and, on an open domain, it is *closed.* The graph $\operatorname{graph}(f) = \{(x, y) : y = f(x)\}$ forms the "floor" of the epigraph. If a sequence of points $(x_k, y_k) \in \operatorname{epi}(f)$ converges to a point on the graph, the limit remains in the epigraph; if it converges to a point strictly above the graph, it is trivially still in the epigraph since the latter is convex and contains every vertical ray above the curve. This closedness yields lower semicontinuity; applying the same idea to $-f$ (whose epigraph is the hypograph of f) gives upper semicontinuity. Having both sides implies continuity at interior points. The concave case is analogous by swapping epigraph with hypograph.