Week 4 - Quasiconvexity and quasiconcavity

Recall (Convex and concave functions)**.** Let $X \subseteq \mathbb{R}^n$ be convex and let $f : X \to \mathbb{R}$.

- f is *convex* if for all $\mathbf{x}, \mathbf{y} \in X$ and all $\alpha \in [0,1]$,

$$f\big(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\big) \;\leq\; \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}).$$

- f is *concave* if for all $\mathbf{x}, \mathbf{y} \in X$ and all $\alpha \in [0,1]$,

$$f\big(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\big) \;\geq\; \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}).$$

Equivalently:

- f is convex $\iff$ its *epigraph*

$$\operatorname{epi}(f) \;=\; \{(\mathbf{x}, y) \in \mathbb{R}^n \times \mathbb{R} :\; y \geq f(\mathbf{x})\} \;\subseteq\; X \times \mathbb{R}$$

  is a convex set.

- f is concave $\iff$ its *hypograph* (a.k.a. subgraph)

$$\operatorname{hypo}(f) \;=\; \{(\mathbf{x}, y) \in \mathbb{R}^n \times \mathbb{R} :\; y \leq f(\mathbf{x})\} \;\subseteq\; X \times \mathbb{R}$$

  is a convex set.

Definition (Quasiconvexity and quasiconcavity)**.** Let $X \subseteq \mathbb{R}^n$ be convex and let $f : X \to \mathbb{R}$.

(1) f is *quasiconvex* if for all $\mathbf{x}, \mathbf{y} \in X$ and all $\alpha \in [0,1]$,

$$f\big(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\big) \;\leq\; \max\{f(\mathbf{x}),\, f(\mathbf{y})\}.$$

(2) f is *strictly quasiconvex* if for all $\mathbf{x} \neq \mathbf{y}$ in X and all $\alpha \in (0,1)$,

$$f\big(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\big) \;<\; \max\{f(\mathbf{x}),\, f(\mathbf{y})\}.$$

(3) f is *quasiconcave* if for all $\mathbf{x}, \mathbf{y} \in X$ and all $\alpha \in [0,1]$,

$$f\big(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\big) \;\geq\; \min\{f(\mathbf{x}),\, f(\mathbf{y})\}.$$

(4) f is *strictly quasiconcave* if for all $\mathbf{x} \neq \mathbf{y}$ in X and all $\alpha \in (0,1)$,

$$f\big(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\big) \;>\; \min\{f(\mathbf{x}),\, f(\mathbf{y})\}.$$

TBW: Look for functions exemplifying all this definition and its combinations.

Proposition (Convex/concave $\Rightarrow$ quasi-(con)vex)**.** *Let $X \subseteq \mathbb{R}^n$ be convex and $f : X \to \mathbb{R}$.*

(1) If f is convex (resp. strictly convex), then f is quasiconvex (resp. strictly quasiconvex).

(2) If f is concave (resp. strictly concave), then f is quasiconcave (resp. strictly quasiconcave).

Proof. (1) Fix $\mathbf{x}, \mathbf{y} \in X$ and $\alpha \in [0,1]$, and set $\mathbf{z}_\alpha = \alpha\mathbf{x} + (1-\alpha)\mathbf{y}$. By convexity,

$$f(\mathbf{z}_\alpha) \le \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \le \alpha \max\{f(\mathbf{x}), f(\mathbf{y})\} + (1-\alpha)\max\{f(\mathbf{x}), f(\mathbf{y})\} = \max\{f(\mathbf{x}), f(\mathbf{y})\},$$

so f is quasiconvex. If f is strictly convex, then for $\mathbf{x} \neq \mathbf{y}$ and $\alpha \in (0,1)$ we have

$$f(\mathbf{z}_\alpha) < \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \le \max\{f(\mathbf{x}), f(\mathbf{y})\},$$

hence f is strictly quasiconvex.

(2) The concave case is analogous. Alternatively, apply part (1) to $-f$: if f is concave, then $-f$ is convex and thus quasiconvex, which is equivalent to f being quasiconcave. The strict variant follows in the same way. □

Remark (Bounds along chords)**.** For any $\mathbf{x}, \mathbf{y} \in X$ and $\alpha \in [0,1]$, with $\mathbf{z}_\alpha = \alpha\mathbf{x} + (1-\alpha)\mathbf{y}$,

$$\max\{f(\mathbf{x}), f(\mathbf{y})\} \ge f(\mathbf{z}_\alpha) \ge \min\{f(\mathbf{x}), f(\mathbf{y})\}.$$

Under strict convexity/concavity, the inequalities are strict whenever $\mathbf{x} \neq \mathbf{y}$ and $\alpha \in (0,1)$.

Remark (Converse fails)**.** The converse of the previous proposition is not always true.

Example (Increasing functions are (strictly) quasi-convex and quasi-concave)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be increasing (resp. strictly increasing). Then f is quasiconvex and quasiconcave (resp. strictly quasiconvex and strictly quasiconcave).

Proof. Fix $x, y \in \mathbb{R}$ with $x > y$ and let $\alpha \in (0,1)$. Set $z_\alpha = \alpha x + (1-\alpha)y$, so $x > z_\alpha > y$. By monotonicity,

$$f(x) \ge f(z_\alpha) \ge f(y) \qquad (\text{resp. } f(x) > f(z_\alpha) > f(y) \text{ if } f \text{ is strictly increasing}).$$

Therefore, for all $\alpha \in (0,1)$,

$$\max\{f(x), f(y)\} \ge f(z_\alpha) \ge \min\{f(x), f(y)\},$$

and in the strict case,

$$\max\{f(x), f(y)\} > f(z_\alpha) > \min\{f(x), f(y)\}.$$

These are precisely the (strict) quasiconvexity and (strict) quasiconcavity inequalities on $\mathbb{R}$.

Definition (Upper and lower contour sets)**.** Let $X \subseteq \mathbb{R}^n$ be nonempty and let $f : X \to \mathbb{R}$. For any $y \in \mathbb{R}$ define

$$U(f,y) := \{\, x \in X : f(x) \ge y \,\} \subseteq X, \qquad L(f,y) := \{\, x \in X : f(x) \le y \,\} \subseteq X.$$

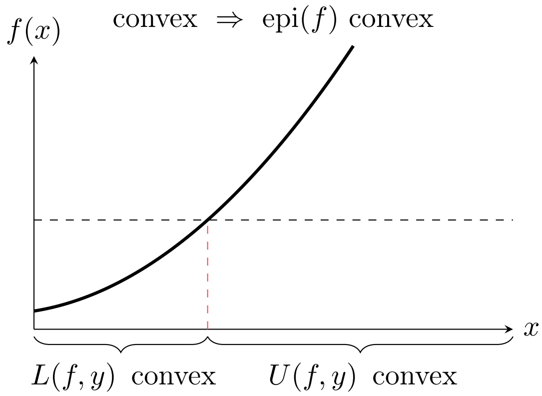
Here $U(f,y)$ is the set of inputs whose value is at least y (hypograph–like at level y), while $L(f,y)$ is the set of inputs whose value is at most y (epigraph–like at level y).

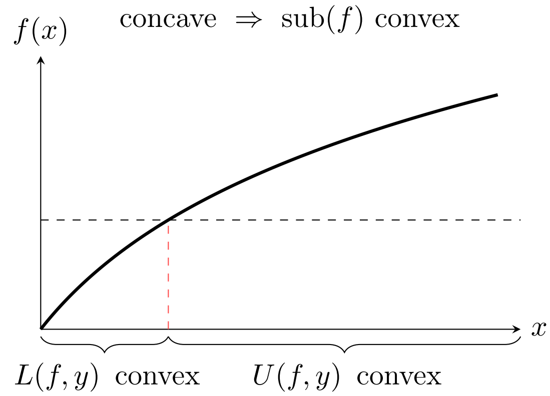
Proposition (Characterization by contour sets)**.** *Let $X \subseteq \mathbb{R}^n$ be convex and $f : X \to \mathbb{R}$.*

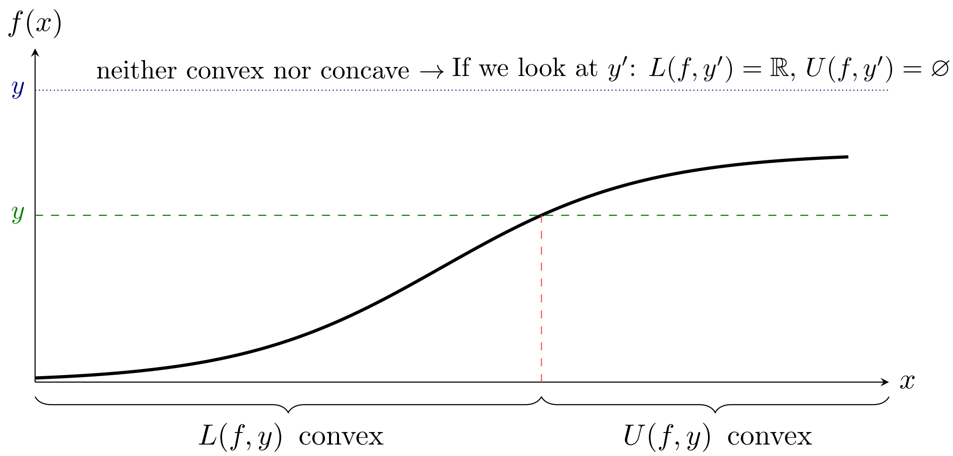
(1) f is quasi-convex $\iff$ *for all $y \in \mathbb{R}$ the lower boundary set*

$$L(f,y) := \{\, x \in X : f(x) \le y \,\}$$

is convex (possibly empty).







(2) f *is* quasiconcave $\iff$ *for all* $y \in \mathbb{R}$ *the upper boundary set*

$$U(f, y) := \{\, x \in X : \; f(x) \geq y \,\}$$

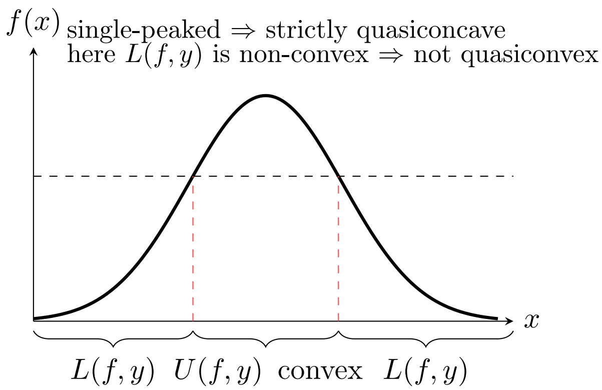
is convex (possibly empty).

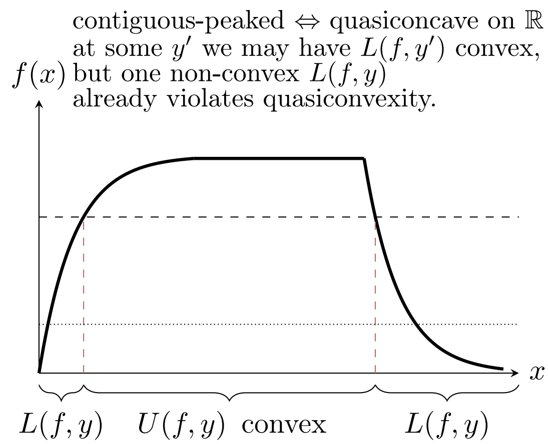
Remark (Clarification on "borderline cases")**.** At some levels y, it may happen that $L(f, y) = X$ or $U(f, y) = X$ (for example, if $y \geq \sup f(X)$, then $L(f, y) = X$, and if $y \leq \inf f(X)$, then $U(f, y) = X$), or that $L(f, y) = \varnothing$ or $U(f, y) = \varnothing$. All these cases satisfy the proposition because X (and, in particular, $\mathbb{R}$ when $X = \mathbb{R}$) is convex and the empty set is also convex.

Example (Single-peaked vs. contiguous-peaked on $\mathbb{R}$; quasi-concavity consequences)**.** Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is *single-peaked* if there exists $m \in \mathbb{R} \cup \{\pm\infty\}$ such that f is strictly increasing on $(-\infty, m)$ and strictly decreasing on (m, ∞). We say that f is *contiguous-peaked* if each upper contour set $U(f, y) = \{x \in \mathbb{R} : f(x) \geq y\}$ is an interval (possibly empty or all of $\mathbb{R}$), allowing for a flat "plateau" of maximizers.

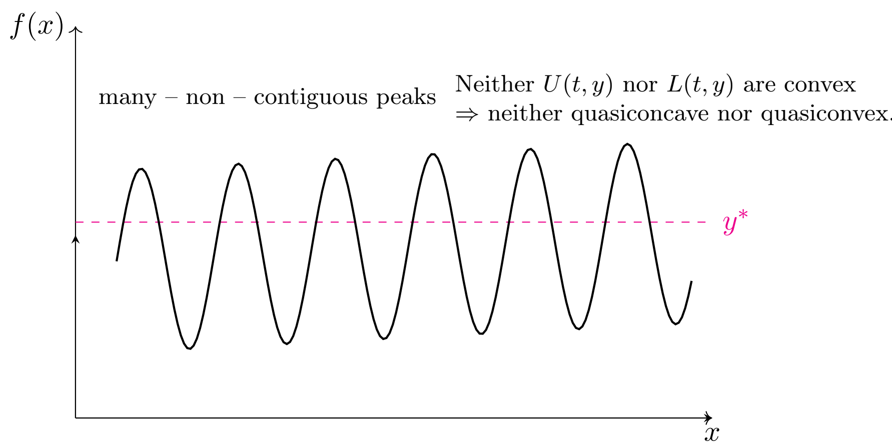
Key facts (on $\mathbb{R}$).

- f is quasiconcave $\iff$ $U(f, y)$ is an interval for every y (i.e., f is contiguous-peaked). Peaks may occur at $\pm\infty$ (monotone functions).
- If f is single-peaked, then f is *strictly* quasiconcave.





Remark (Why a single convex $L(f,y')$ does not suffice)**.** Quasiconvexity requires *every* lower contour $L(f,y)$ to be convex. In the single-peaked panel, for the displayed y the set $L(f,y)$ is the union of two disjoint intervals (non-convex), so f fails quasiconvexity—even though for some other levels y' the set $L(f,y')$ happens to be convex.



Punchline. On $\mathbb{R}$:

- One contiguous peak (possibly a flat plateau) $\iff$ f is quasiconcave (equivalently, each $U(f,y)$ is an interval). The "peak" may occur at $+\infty$ or $-\infty$ (monotone functions).
- If f is single-peaked (strictly increasing up to a unique mode m and strictly decreasing thereafter—allowing $m=\pm\infty$), then f is *strictly* quasiconcave.