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Contents

Class 1: Consumer Choice, Preferences, and Rationalization

From Utility Maximization to Choice Correspondences

A consumer chooses a bundle $x = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell_+$ to

$$\max_{x \in \mathbb{R}^\ell_+} u(x) \quad \text{s.t.} \quad p \cdot x \leq y,$$

where $p \in \mathbb{R}^\ell_{++}$ is the price vector and $y > 0$ income.

Definition (Ordinal utility & monotone transforms)**.** A function $u : X \to \mathbb{R}$ *represents* a preference relation $\succeq$ on X if

$$x \succeq y \iff u(x) \geq u(y) \quad \text{for all } x, y \in X.$$

If $f : \mathbb{R} \to \mathbb{R}$ is *strictly increasing*, then $v = f \circ u$ represents the same $\succeq$. Hence utility is *ordinal*: only the induced ranking matters.

Example (Log-transform of Cobb–Douglas)**.** If $u(x_1, x_2) = x_1 x_2$, then $v(x_1, x_2) = \ln x_1 + \ln x_2 = f(u)$ with $f(t) = \ln t$ (strictly increasing on $t > 0$) represents the same preferences.

Definition (Feasible sets and choice)**.** Let X be the set of feasible objects and $\mathcal{A}$ a family of nonempty subsets $A \subseteq X$ ("menus"). A *choice correspondence* is a mapping

$$c : \mathcal{A} \rightrightarrows X, \qquad c(A) \subseteq A,$$

assigning to each menu the set of chosen elements.

We use correspondences (possibly multi-valued) because data/behavior may show ties.

Quick read and intuition. X is the *universe* of feasible objects (all alternatives that could be considered). A *menu* $A \subseteq X$ is the subset actually on the table in a decision instance (what is "on offer"). The choice correspondence c takes each menu A and returns the set of alternatives actually chosen when A is offered. The double arrow $c : \mathcal{A} \rightrightarrows X$ reminds us that c may be *multi-valued*.

"No off-menu": the condition $c(A) \subseteq A$ forbids choosing something that was not available.

Why a correspondence *and not a function?* Two main reasons:

- **Tie/indifference**: if the person is indifferent between x and y in A, the model does not impose a tie-break, so $c(A)$ may include both.
- **Repeated data**: if we observe several decisions with the same A and sometimes x is chosen and other times y, it is natural to record $c(A) = \{x, y\}$.

If there is always a unique chosen element, c is called a choice *function*.

Link to utility. If there exists a preference $\succeq$ (or a utility u that represents it), a standard way to rationalize c is

$$c(A) = \arg\max_{x \in A} u(x).$$

Ties in $u \Rightarrow c(A)$ may contain multiple elements. With strictly convex preferences over convex sets (e.g., $\mathbb{R}^{\ell}_{+}$ with Cobb–Douglas) the arg max is unique; with discrete domains or preferences with flat segments, multiple optima may appear.

Non-emptiness is often imposed: $c(A) \neq \varnothing$ for every $A \in \mathcal{A}$. In finite domains this is reasonable; in infinite ones it requires continuity/compactness assumptions for a maximum to exist.

Domain of c. $\mathcal{A}$ is the family of menus on which we define/observe choice. In revealed-preference theory we often take *all* nonempty subsets of X; with real data, $\mathcal{A}$ consists only of the menus that were observed.

Example: Discrete menus and budget sets.

Discrete. Let $X = \{\text{Air}, \text{ThinkPad}, \text{XPS}\}$ and suppose we observe:

$$\begin{aligned}
&c(\{\text{Air}, \text{ThinkPad}\}) = \{\text{Air}, \text{ThinkPad}\} \qquad \text{(sometimes Air is chosen, sometimes ThinkPad)},\\
&c(\{\text{ThinkPad}, \text{XPS}\}) = \{\text{ThinkPad}\},\\
&c(\{\text{Air}, \text{XPS}\}) = \{\text{Air}\}.
\end{aligned}$$

Here c is a *correspondence* because for the first menu there is a tie and we record both choices. If later the menu is $\{\text{Air}, \text{ThinkPad}, \text{XPS}\}$, coherence (WARP) requires not choosing ThinkPad while Air is available if Air was ever chosen over ThinkPad.

Budget set: Let $X = \mathbb{R}^2_+$ and $A = B(p, y) = \{x \in \mathbb{R}^2_+ : p_1 x_1 + p_2 x_2 \leq y\}$.

- Cobb–Douglas. For $u(x) = x_1^{\alpha} x_2^{1-\alpha}$ with $\alpha \in (0, 1)$, the FOCs yield

  $$x_1^* = \frac{\alpha}{p_1} y, \qquad x_2^* = \frac{1-\alpha}{p_2} y.$$

  Hence $c(A) = \{(x_1^*, x_2^*)\}$ is *unique*. Strict convexity of preferences guarantees no ties.

- Perfect complements (Leontief). For $u(x) = \min\{x_1, x_2\}$ the optimum occurs at the kink $x_1 = x_2$ (consumption in 1:1 proportion) subject to the budget:

$$x_1^* = x_2^* = \frac{y}{p_1 + p_2}.$$

  Here too $c(A)$ is *unique*: the indifference "L" has a single intersection with the budget line when $p_1, p_2 > 0$.

- Perfect substitutes. For $u(x) = x_1 + x_2$:

$$c(A) = \begin{cases} \{(y/p_1, 0)\}, & \text{if } p_1 < p_2 \quad \text{(corner at } x_1\text{)}, \\ \{(0, y/p_2)\}, & \text{if } p_2 < p_1 \quad \text{(corner at } x_2\text{)}, \\ \{x \in \mathbb{R}^2_+ : \ p_1x_1 + p_2x_2 = y\}, & \text{if } p_1 = p_2 \quad \text{(all points on the segment)}. \end{cases}$$

  This last case illustrates why we model c as a *correspondence*: when the slope of the budget line matches that of indifference curves, there are *infinitely many* optima.

In continuous domains $c(A)$ can be a singleton (strictly convex preferences or "point" kinks) or multivalued (flat segments or slope coincidences). In discrete data, c becomes multivalued when we observe ties or different choices under the same menu.

Preferences and Their Basic Properties

Definition (Preference primitives)**.** A *weak* preference $\succeq$ is a binary relation on X. Define the *strict* preference $x \succ y$ iff $x \succeq y$ and not $y \succeq x$, and *indifference* $x \sim y$ iff $x \succeq y$ and $y \succeq x$.

Definition (Rational (standard) axioms)**.** A preference $\succeq$ is *complete* if for all $x, y \in X$ we have $x \succeq y$ or $y \succeq x$ (or both); and *transitive* if $x \succeq y$ and $y \succeq z$ imply $x \succeq z$.

Remark. Completeness formalizes "the agent can compare any two options"; transitivity formalizes *consistency* of those comparisons. With these, the derived $\sim$ is an equivalence relation.

From Choice Data to Preferences: Coherence (WARP)

Kreps call it *choice coherence*. In standard revealed-preference language it is Samuelson's *Weak Axiom of Revealed Preference* (WARP).

Definition (Non-emptiness)**.** c satisfies *non-emptiness* if $c(A) \neq \varnothing$ for every $A \in \mathcal{A}$.

Definition (Choice coherence = WARP)**.** c satisfies *choice coherence* (WARP) if for all menus $A, B \in \mathcal{A}$ and all $x, y \in A \cap B$:

$$\big(x \in c(A) \text{ and } y \in A \setminus c(A)\big) \implies y \notin c(B).$$

Interpretation: If x is chosen over y when both were available, then y is never chosen in any other menu that still contains x.

Your red sketch labeled "violation of choice coherence" is exactly a WARP violation.

Definition (Rationalization by a preference)**.** A preference $\succeq$ *rationalizes* c if for every $A \in \mathcal{A}$,

$$c(A) = \{x \in A : x \succeq y \text{ for all } y \in A\}.$$

That is, $c(A)$ is the set of $\succeq$-maximal elements in A.

Key Equivalences on Finite Domains

Throughout this section assume X is finite (as in our class; Kreps develops the infinite case).

Theorem (Revealed preference characterization on finite X)**.** *Let X be finite and c defined on all nonempty $A \subseteq X$. Then the following are equivalent:*

1. *c satisfies non-emptiness and choice coherence (WARP).*
2. *There exists a* complete and transitive *preference $\succeq$ that rationalizes c (i.e., $c(A)$ are $\succeq$-maxima in A for all A).*

Proof sketch your professor outlined. (2) $\Rightarrow$ (1): If $c(A)$ are the $\succeq$-maximal elements in each A, then $c(A) \neq \varnothing$ (finite sets have maxima under complete, transitive $\succeq$), and WARP holds because if $x \succeq y$ in A with $y \notin c(A)$, then in any B containing both, y cannot beat x.

(1) $\Rightarrow$ (2) (construction): Define a *revealed weak preference* by

$$x \succeq^* y \iff x \in c(\{x, y\}).$$

Completeness: Because $c(\{x, y\}) \neq \varnothing$ for every pair, either x or y is chosen from $\{x, y\}$, hence either $x \succeq^* y$ or $y \succeq^* x$.

This answers your "Why?": Non-emptiness on all pairs forces a choice between any two options, which is exactly completeness of $\succeq^*$.

Transitivity: Suppose $x \succeq^* y$ and $y \succeq^* z$. If transitivity failed ($x \not\succeq^* z$), then $z \in c(\{x, z\})$. Consider menu $A = \{x, y, z\}$. WARP applied to the pairs $\{x, y\}$ and A forces $y \notin c(A)$; WARP applied to $\{y, z\}$ and A forces $z \notin c(A)$. But then $c(A)$ would be empty, contradicting non-emptiness. Hence $x \succeq^* z$.

Finally, verify that $c(A)$ equals the set of $\succeq^*$-maximal elements of A (otherwise WARP would be violated by the offending pair inside A). $\square$

Corollary (Utility representation on finite X)**.** *If $\succeq$ is complete and transitive on finite X, then there exists a utility $u : X \to \mathbb{R}$ that represents $\succeq$.*

For instance, assign u by ranks: largest u to any $\succeq$-maximal elements, then proceed recursively on the remainder. Any strictly increasing transform $f \circ u$ represents the same $\succeq$.

Linking Back to Utility Maximization

Proposition (Utility maximization rationalizes c)**.** *Fix a preference $\succeq$ represented by u. Define $c(A) = \arg\max_{x \in A} u(x)$. Then c satisfies non-emptiness and WARP, and c is rationalized by $\succeq$.*

Remark (When do we need more structure (continuity, convexity)?)**.** For *infinite* X (e.g., $X = \mathbb{R}^{\ell}_{+}$), completeness and transitivity alone do not guarantee a *continuous* or *nice* u. Standard micro adds *continuity* (to rule out jumps), *monotonicity* ("more is better"), and sometimes (strict) *convexity* (diminishing marginal rate of substitution). In the finite case of this class, Theorem and the ranking construction suffice.

Quick Checks & Exercises

Exercise 0.0.1 (Diagnose a coherence (WARP) violation)**.** Suppose $c(\{x, y\}) = \{x\}$ and $c(\{x, y, z\}) = \{y\}$. Show this violates WARP. Which pair produces the contradiction?

Exercise 0.0.2 (Reconstructing $\succeq$ from c)**.** Given choice data on all nonempty $A \subseteq X$ for finite X, build $\succeq$ as in the proof and show $c(A)$ equals the set of $\succeq$-maximal elements of A for every A.

Exercise 0.0.3 (Monotone transforms)**.** Let u represent $\succeq$ on $X \subseteq \mathbb{R}^{\ell}_{+}$. Prove that $v = f \circ u$ represents $\succeq$ if and only if f is strictly increasing on $u(X)$.

Exercise 0.0.4 (Utility from *any* complete, transitive $\succeq$ on finite X)**.** Construct a representing u by top-cycling ranks: set $u(x) = k$ for all x in the k-th indifference "layer" in the order from best to worst. Argue that this works and is unique up to strictly increasing transforms.

Mini FAQ

Q: Why does non-emptiness on all pairs imply completeness? Because for each $\{x, y\}$ the choice set $c(\{x, y\})$ contains either x or y. Declare $x \succeq y$ iff $x \in c(\{x, y\})$. Then for every pair, at least one direction holds—exactly completeness.

Q: Is choice coherence *the same as WARP?* Yes in this finite, menu-based setting: "If x was chosen over y somewhere, never choose y when

x is also available."

Class 2: Rationality of Choice (finite and infinite)

Quick recap (finite case)

Let X be the set of alternatives (finite in this subsection) and let $\mathcal{A} \subseteq 2^X \setminus \{\varnothing\}$ be the family of feasible menus.

Definition (Choice function)**.** A *choice function* is a mapping $c : \mathcal{A} \to 2^X$ such that $c(A) \subseteq A$ for every $A \in \mathcal{A}$. We say that c satisfies *nonemptiness* if $c(A) \neq \varnothing$ for all $A \in \mathcal{A}$.

Definition (Preferences and utility)**.** A *preference relation* is a complete and transitive binary relation $\succeq \subseteq X \times X$. Write $x \succ y$ for strict preference and $x \sim y$ for indifference. A *utility function* is a map $u : X \to \mathbb{R}$.

Definition (Rationalization and representation)**.** We say that $\succeq$ *rationalizes* c if, for every $A \in \mathcal{A}$,

$$c(A) = \{\, x \in A : \ x \succeq y \ \ \forall y \in A \,\}.$$

We say that u *represents* $\succeq$ if $u(x) \geq u(y) \iff x \succeq y$ for all $x, y \in X$.

Consistency of choice (WARP). We use the operational form

(WARP) If $x, y \in A \cap B$, $x \in c(A)$ and $y \in c(B)$, then $x = y$.

Equivalently: if $x, y \in A$, $x \in c(A)$, $y \notin c(A)$, and $x \in B$, then $y \notin c(B)$.[1]

[1] *Intuition for future me:* once x has been chosen over y when both were available, y cannot "beat" x in any other menu where both remain present.

Proposition (Rationalization on finite X)**.** *There exists a complete and transitive preference relation $\succeq$ that rationalizes the choice function c* if and only if *the choice function c satisfies nonemptiness and WARP.*

Proof idea. ($\Rightarrow$) If c comes from maximizing $\succeq$ in each menu, a maximal element always exists (nonemptiness), and no "revenge" by y over x is possible when $x \succeq y$ (WARP).

($\Leftarrow$) Build $\succeq$ from pairwise choices:

$$x \succeq y \quad \iff \quad x \in c(\{x, y\}).$$

We want, for all A,

$$x \in c(A) \iff x \succeq y \ \ \forall y \in A.$$

($\Rightarrow$) By *contraposition*: if $x \in c(A)$ but $\exists y \in A$ with $x \notin c(\{x,y\})$, then $c(\{x,y\}) = \{y\}$. WARP forbids x being chosen in A with y available—contradiction.

($\Leftarrow$) If $x \succeq y$ for all $y \in A$, then x wins every pairwise contest in A. If $x \notin c(A)$, nonemptiness gives some $y \in c(A)$. But WARP, applied to $\{x,y\}$ and A, forbids choosing y in A when x was available and also wins the binary comparison—contradiction. □

Proposition (Utility on finite X)**.** *If $\succeq$ is complete and transitive on finite X, there exists $u : X \to \mathbb{R}$ representing $\succeq$.* Construction: *rank X and assign increasing values (e.g., $u = 3, 2, 1, \ldots$).*

Proposition (Invariance to monotone transformations)**.** *If u represents $\succeq$ and $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing on the range of u, then $f \circ u$ also represents $\succeq$.*

"step-by-step" proof of Proposition

Define the preference relation $\succeq$ via binary choices: $x \succeq y \iff x \in c(\{x,y\})$. We seek, for all A:

$$x \in c(A) \iff x \in c(\{x,y\}) \ \forall y \in A.$$

(i) $\Rightarrow$ If $x \in c(A)$ but $\exists y \in A$ with $x \notin c(\{x,y\})$, then $c(\{x,y\}) = \{y\}$. By WARP, y cannot be chosen in any menu where x is present. Since $y \in A$ and $x \in A$, this contradicts $x \in c(A)$.

(ii) $\Leftarrow$ Suppose $x \notin c(A)$. By nonemptiness, $\exists y \in c(A)$. Since $x \succeq y$ by hypothesis (it wins every binary), we have $x \in c(\{x,y\})$. Applying WARP between $\{x,y\}$ and A, y could not be chosen in A. Contradiction.

Intuition comment. Part (ii) formalizes: "if x defeats everyone head-to-head, it must be the tournament champion."[2]

[2] *Intuition for future me:* with finite X, WARP kills cycles and ensures the *binary maximum* is a menu maximum.

What if X is infinite? Why things can fail without extra axioms

Example. Let $X = \mathbb{R}^2_+$ and $u(x_1, x_2) = x_1 + x_2$. On the "unrestricted" menu $A = X$, there is no maximizer: $c(X) = \varnothing$ (you can always increase a bit more).

To recover a useful notion of rationality on infinite sets, we impose three minimal conditions on c:

(i) Finite nonemptiness: if A is finite and nonempty, then $c(A) \neq \varnothing$.

(ii) Choice cohoerency (WARP) .

(iii) Binary consistency (pairwise maximality): if $x \in A$ and $x \in c(\{x, y\})$ for every $y \in A$, then $x \in c(A)$.

Proposition (Rationalization on infinite X)**.** *If c satisfies (i)–(iii), then there exists a complete and transitive $\succeq$ that rationalizes c (constructed from binary choices as above).*

Idea. (i)–(ii) ensure the induced binary relation is consistent; (iii) promotes the "pairwise champion" to menu champion even when A is infinite.[3]

[3] *Intuition for future me:* (iii) is exactly the missing piece when you passed from pairs to large menus; it formalizes "if x wins locally against every rival, it must win globally."

Proposition (Utility representation on infinite X (countable order-dense criterion))**.** *Let $\succeq$ be a* weak order *(complete and transitive) on X. There exists $u : X \to \mathbb{R}$ that represents $\succeq$* iff *there is a countable subset $X^* \subseteq X$* ***order-dense*** *such that*

$$\forall x \succ y \;\; \exists x^* \in X^* \textit{ with } \; x \succeq x^* \succ y.$$

Intuition. In $(\mathbb{R}, \geq)$, $\mathbb{Q}$ is countable and *dense*: if $x > y$, you can insert a rational q with $x \geq q > y$. That property lets us "number" the order with reals without big jumps and build u. Then Proposition ensures any strictly increasing transform preserves $\succeq$.[4]

[4] *Intuition for future me:* order-density is the substitute for finiteness: "I can always slip in a numeric marker between any two strictly ordered alternatives."

Class 3: Consumer Choice and Demand (Kreps Ch. 2)

Primitives and the baseline assumptions

We work with k goods and the *consumption set* $X = \mathbb{R}^k_+$. A bundle is $x = (x_1, \dots, x_k) \in X$.

Intuition for future me. In this class, we **only** assume completeness and transitivity. In infinite domains, this *does not* guarantee maximizers or continuous utility representation. Several conclusions from the finite section do not automatically apply.

Definition (Preferences on X)**.** A (weak) preference relation $\succeq \subseteq X \times X$ is assumed *complete and transitive*. Write

$$x \succ y \iff \big(x \succeq y \text{ and not } y \succeq x\big), \qquad x \sim y \iff \big(x \succeq y \text{ and } y \succeq x\big).$$

Definition (Utility representation (not assumed a priori))**.** A function $u : X \to \mathbb{R}$ *represents* $\succeq$ if $u(x) \geq u(y) \iff x \succeq y$.

Budget sets and (Walrasian) demand

Intuition for future me. d is (again) a *correspondence*: it can be multivalued or even empty if there are no maxima (e.g., preferences without continuity/compactness). With $p \gg 0$, $B(p, y)$ is a bounded polytope in $\mathbb{R}^k_+$.

Let prices be $p \in \mathbb{R}^k_{++}$ and income $y > 0$. The *budget set* is

$$B(p, y) := \{\, x \in X : \ p \cdot x \leq y \,\}.$$

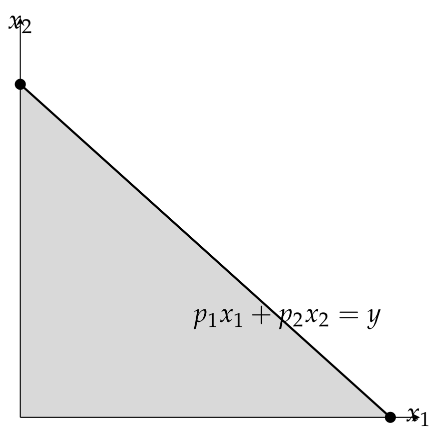


Figure 1: Budget set $B(p, y)$ in $k = 2$: region under the budget line.

Definition (Demand correspondence)**.** The (Walrasian) demand at (p, y) is the set of $\succeq$-maximal elements in the budget set:

$$d(p, y) := \big\{\, x \in B(p, y) : \ x \succeq x' \text{ for all } x' \in B(p, y) \big\}.$$

Indifference "curves" versus mere points

With additional structure, indifference sets typically look like smooth, downward-sloping, convex curves. Without such assumptions we cannot guarantee any shape—empirically we might only "see" scattered points.

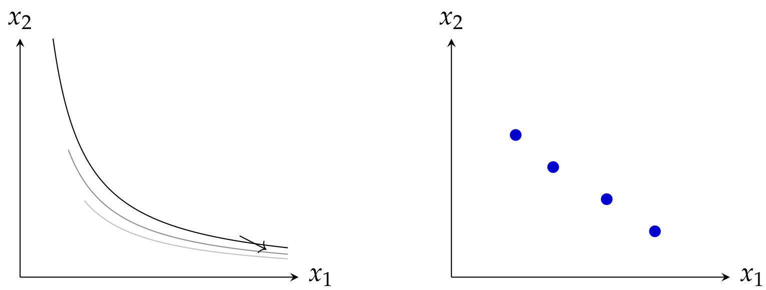


Figure 2: Left: "indifference curves" under regularity. Right: without assumptions, we may only observe a set of points.

Intuition for future me. Indifference curves emerge from structural assumptions (continuity, monotonicity, convexity). Without them, the set $\{x : x \sim \bar{x}\}$ can be pathological or disconnected.

Structural properties (preferences and utility)

Definition (Monotonicity)**.** A preference $\succeq$ on $\mathbb{R}^k_+$ is *monotone* if for any $x, x' \in \mathbb{R}^k_+$,

$$x_i \geq x'_i \text{ for all } i = 1, \dots, k \quad \Longrightarrow \quad x \succeq x'.$$

Equivalently, "more doesn't hurt you" componentwise.

Definition (Strict/strong monotonicity)**.** $\succeq$ is *strictly* (a.k.a. *strongly*) monotone if the above holds and, whenever $x_i \geq x'_i$ for all i and $x_j > x'_j$ for some j, then $x \succ x'$.

Intuition for future me. Under monotonicity and $p \gg 0$, any optimum $x^* \in d(p, y)$ must exhaust the budget: $p \cdot x^* = y$. Otherwise, you could increase a good a little bit and improve (or not worsen). ICs are nonincreasing and the NE quadrant is "better".

Remark. Together with mild continuity/convexity, monotonicity delivers well-behaved indifference sets and existence/uniqueness results for $d(p, y)$; we will call on them when needed below.

Monotonicity $\leftrightarrow$ (non)decreasing utility

Definition (Strict/strong monotonicity)**.** A preference $\succeq$ is *strictly* (a.k.a. *strongly*) monotone if

$$x \geq x' \text{ and } x \neq x' \quad \Longrightarrow \quad x \succ x'.$$

Proposition (Equivalence at the utility level)**.** *Assume $\succeq$ admits a utility representation (no continuity required).*[5]

(a) $\succeq$ *is* monotone $\Longleftrightarrow$ every *utility u representing* $\succeq$ *is* nondecreasing *in the product order:*

$$x \geq x' \implies u(x) \geq u(x').$$

[5] *Intuition for future me:* Under the product order, "more of each good never hurts." If a representing u decreased in some component, it could invert $x \geq x'$ rankings. The statement uses that strictly increasing transforms preserve $\succeq$.

(b) *A preference relation* $\succeq$ *is* strictly monotone $\iff$ every *representing utility* u *is* strictly increasing*:*

$$x \geq x',\ x \neq x' \implies u(x) > u(x').$$

Insatiability: global vs. local

Definition (Global insatiability (GI))**.** A preference relation $\succeq$ is *globally insatiable* if

$$\forall x \in X\ \exists x' \in X \text{ such that } x' \succ x.$$

Definition (Local insatiability (LI))**.** A preference relation $\succeq$ is *locally insatiable* [6] if

$$\forall x \in X,\ \forall \varepsilon > 0\ \exists x' \in X \text{ with } \|x' - x\| \leq \varepsilon \text{ and } x' \succ x.$$

[6] *Intuition for future me.* **GI**: for every bundle x there is *some* x' (possibly far) with $x' \succ x$. **LI**: for every x and every $\varepsilon > 0$ there is x' within ε of x with $x' \succ x$—an improvement *arbitrarily close*.

Remark. LI $\Rightarrow$ GI (take, e.g., $\varepsilon = 1$). The converse fails: one may always find some improvement, but only far away.

Example (GI without monotonicity)**.** On $X = \mathbb{R}^2_+$, $u(x_1, x_2) = x_1 - x_2$ yields GI: for any x, $(x_1 + 1, x_2) \succ x$. But preferences are not monotone (raising x_2 makes you worse off).

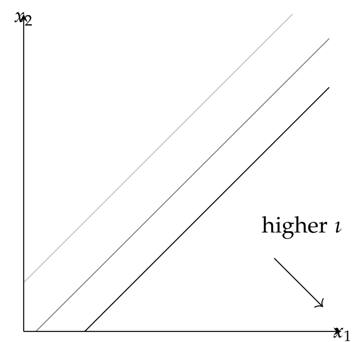


Figure 3: GI without monotonicity: $u = x_1 - x_2$.

Remark (LI and budget exhaustion)**.** If $p \gg 0$, $y > 0$ and $\succeq$ is LI, then any $x^* \in d(p, y)$ satisfies $p \cdot x^* = y$. *Idea:* if $p \cdot x^* < y$, LI yields x' arbitrarily close with $x' \succ x^*$; for small ε, $p \cdot x' \leq y$, contradicting optimality.

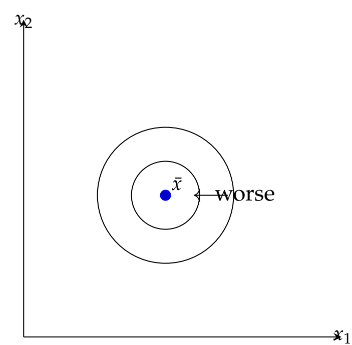


Figure 4: Local satiation (bliss point): LI fails at $\bar{x}$.

Convexity and upper contour sets

Definition (Convex preferences)**.** A preference relation $\succeq$ is *convex* if for every $z \in X$ the weakly preferred set (this is slightly different than the upper contour set, right?)

$$U(z) := \{x \in X :\ x \succeq z\} = NWT(z)$$

is convex. *NWT* states for "not worse than" in Kreps notation. Equivalently: if $x \succeq z$ and $y \succeq z$, then $\alpha x + (1 - \alpha) y \succeq z$ for all $\alpha \in [0, 1]$.

Proposition (Characterization via $U(z)$)**.** *A preference relation* $\succeq$ *is convex* $\iff$ *each* $U(z)$ *is convex.*

Proposition (Quasiconcave representation)**.** *If* $\succeq$ *is convex and has a utility representation, then there exists a representing utility* u *that is* quasiconcave *(its upper contour sets* $\{x :\ u(x) \geq t\}$ *are convex).*

Intuition: Convexity encodes "love for averages": if two bundles are at least as good as z, any mixture is too. That is exactly the convex-upper-set property of a quasiconcave u.

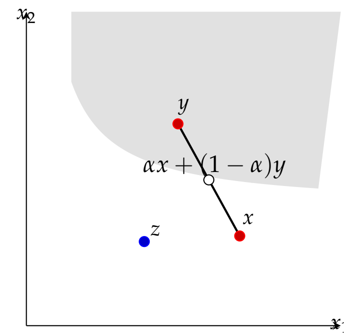


Figure 5: Convexity: $U(z)$ convex (mixtures no worse than z).

Concavity vs. convex preferences

Proposition (Concave utility $\Rightarrow$ convex preferences)**.** *If a preference relation $\succeq$ is represented by a* concave *utility u, then the preference relation $\succeq$ is convex.*

One-line proof idea. For any z, the upper contour set $U(z) = \{x : u(x) \geq u(z)\}$ is convex because for $x, y \in U(z)$ and $\alpha \in [0,1]$,

$$u(\alpha x + (1-\alpha)y) \;\geq\; \alpha u(x) + (1-\alpha)u(y) \;\geq\; u(z),$$

so $\alpha x + (1-\alpha)y \in U(z)$. By Prop. , the preference relation $\succeq$ is convex.[7] $\square$

[7] *Intuition for future me:* Concavity gives "diminishing marginal utility." Superlevel sets of a concave map are convex, hence mixtures are no worse than z—exactly convex preferences.

Remark (The converse need not hold)**.** Convex preferences admit a *quasiconcave* representation (Prop.), but *not every* representing utility is concave. E.g., if u is concave, then $v = \exp(u)$ is strictly increasing (represents the same order) but typically *not* concave. Quasiconcavity is the right ordinal notion.

Quasiconcavity and transformations

Definition (Quasiconcavity)**.** A function $u : X \to \mathbb{R}$ is *quasiconcave* if for all $x, x' \in X$ and $\alpha \in [0,1]$,

$$u(\alpha x + (1-\alpha)x') \;\geq\; \min\{u(x), u(x')\}.$$

Equivalently, every *upper level set* $\{x : u(x) \geq c\}$ is convex.

Proposition (Convex preferences $\iff$ quasiconcave utility)**.** *Suppose $\succeq$ has a utility representation. Then $\succeq$ is convex* iff every *utility representing $\succeq$ is quasiconcave.*

Why "every"? If u is quasiconcave and f is strictly increasing, then $v = f \circ u$ has the same upper sets up to a relabeling of thresholds:

$$\{x : v(x) \geq c\} = \{x : u(x) \geq f^{-1}(c)\},$$

so v is also quasiconcave. Since any two utilities that represent the same weak order differ by a strictly increasing transform, the property is invariant across *all* representatives. $\square$

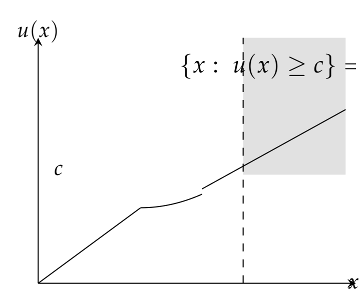


Figure 6: Not concave but quasiconcave in 1D: upper sets are intervals.

Remark (Handy facts)**.** To remember:

- In one dimension ($X \subseteq \mathbb{R}$), *every increasing function is quasiconcave* (upper sets are rays $[a, \infty)$).
- Concavity $\Rightarrow$ quasiconcavity, but not conversely.

Choice from convex sets under convex preferences

For any feasible set $A \subseteq X$, define the set of (weak) optima

$$C_{\succeq}(A) := \{\, x \in A : \; x \succeq x' \text{ for all } x' \in A \,\}.$$

Proposition ($C_{\succeq}(A)$ is convex)**.** *If A is convex and $\succeq$ is convex, then $C_{\succeq}(A)$ is convex.*

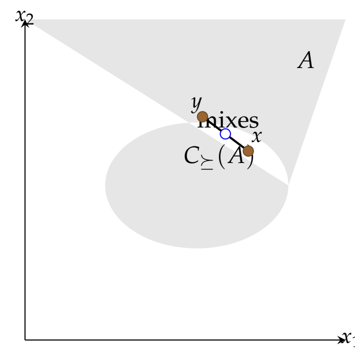


Figure 7: With convex $\succeq$ and convex A, the argmax set is convex.

Proof sketch. Take $x, y \in C_{\succeq}(A)$. For any $x' \in A$, we have $x \succeq x'$ and $y \succeq x'$. By convexity of $\succeq$,

$$\alpha x + (1 - \alpha) y \;\succeq\; x' \qquad \forall \alpha \in [0, 1].$$

Since A is convex, $\alpha x + (1 - \alpha) y \in A$; hence it is weakly preferred to every element of A and thus belongs to $C_{\succeq}(A)$.[8] □

[8] *Intuition for future me:* If two bundles beat every feasible rival, any average of them also beats every rival (love for averages). Therefore the whole "best set" is a convex set.

Class 4: Regularity of Preferences

Roadmap

Preferences are the primitives. When they satisfy regularity properties, *some* (ordinally meaningful) properties can be imposed on a representing utility as well. We work on $X = \mathbb{R}^k_+$, with a complete and transitive $\succeq$.

Convexity (love for averages)

Definition (Convex preferences)**.** A preference relation $\succeq$ is *convex* if for every $z \in X$, the weak upper contour set

$$U(z) := \{x \in X : \; x \succeq z\}$$

is convex. Equivalently: if $x \succeq z$ and $y \succeq z$, then $\alpha x + (1-\alpha)y \succeq z$ for all $\alpha \in [0,1]$.[9]

[9] *Intuition for future me:* "Love for averages": if two bundles are no worse than z, any mixture is also no worse than z. Upper contour sets look convex (bulging toward the origin).

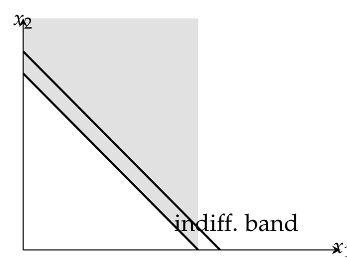


Figure 8: Convex prefs with a *fat* indifference set: convex ✓, semi–strict ×.

Semi–strict and strict convexity

There are three common strengths (chains hold left $\Rightarrow$ right):

$$\text{strictly convex} \;\Rightarrow\; \text{semi–strictly convex} \;\Rightarrow\; \text{convex.}$$

Definition (Semi–strict convexity)**.** A preference relation $\succeq$ is *semi–strictly convex* if they *are convex* and for all x, y with $x \succeq y$ and $x \neq y$,

$$\alpha x + (1-\alpha)y \succ y \qquad \forall\, \alpha \in (0,1).$$

Mixing something *weakly better* with y makes you *strictly* better than y.

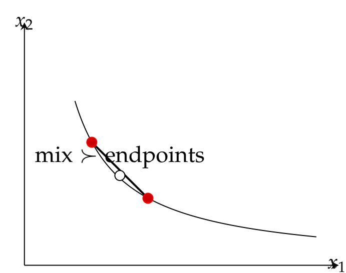


Figure 9: Strict convexity: the chord lies *above* the IC $\Rightarrow$ mixtures strictly better.

Definition (Strict convexity)**.** A preference relation $\succeq$ is *strictly convex* if for all $x \neq y$ with $x \succeq y$,

$$\alpha x + (1-\alpha)y \succ y \qquad \forall\, \alpha \in (0,1).$$

In particular, if $x \sim y$ and $x \neq y$, every mixture strictly improves on y.

Remark (Semi–strict + continuity $\Rightarrow$ convex)**.** If the preference relation $\succeq$ is *continuous* (next section), semi–strict convexity implies convexity. Intuition: the one–sided strict improvement along chords, plus closed upper sets, forces the whole upper set to be convex.

Add plot of not strictly convex preferences.

Proposition (Uniqueness of demand with strict convexity)**.** *If $A \subseteq X$ is convex (e.g., a budget set) and $\succeq$ is strictly convex, then $C_{\succeq}(A)$ is a* singleton*. With LI or monotonicity and $p \gg 0$, the maximizer lies on $p \cdot x = y$.*

Quasiconcavity (QC), semi–strict QC, and strict QC

Definition (Quasiconcavity (QC))**.** A function $u : X \to \mathbb{R}$ is *quasiconcave* if for all $x, y \in X$ and $\alpha \in [0, 1]$,

$$u(\alpha x + (1-\alpha)y) \;\geq\; \min\{u(x), u(y)\}.$$

Equivalently, every upper contour set $\{x : \; u(x) \geq c\}$ is convex.

Definition (Semi–strict quasiconcavity)**.** A function $u : X \to \mathbb{R}$ is *semi–strictly quasiconcave* if whenever $u(x) > u(y)$ and $\alpha \in (0, 1)$,

$$u(\alpha x + (1-\alpha)y) \;>\; u(y).$$

Mixing a strictly better point with a worse one gives a strict improvement over the worse one.

Definition (Strict quasiconcavity)**.** A function $u : X \to \mathbb{R}$ is *strictly quasiconcave* if for all $x \neq y$ and $\alpha \in (0, 1)$,

$$u(\alpha x + (1-\alpha)y) \;>\; \min\{u(x), u(y)\}.$$

Equivalently: if $u(x) = u(y)$ with $x \neq y$, then any strict mixture has strictly *higher* utility.

Proposition (Invariance under monotone transforms)**.** *If u is (semi–strictly / strictly) quasiconcave and $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $f \circ u$ is also (semi–strictly / strictly) quasiconcave.*

Proposition (Preferences vs. QC of utilities)**.** *Assume $\succeq$ is represented by some u.*

(a) *$\succeq$ is convex $\iff$* every *representing utility is quasiconcave.*

(b) *If u is strictly quasiconcave, then $\succeq$ is* strictly *convex.*

(c) *If $\succeq$ is strictly convex and continuous, then there exists a representing utility that is strictly quasiconcave.*

Proposition (Maximizers on convex sets)**.** *Let $A \subseteq X$ be convex.*

(a) If u is quasiconcave, the maximizer set $\arg\max_{x\in A} u(x)$ is convex.

(b) If u is strictly *quasiconcave, then $\arg\max_{x\in A} u(x)$ is a singleton.*

Why concavity ⇒ quasiconcavity (but not conversely)? Concavity imposes the stronger Jensen inequality:

$$u(\alpha x + (1-\alpha)y) \;\geq\; \alpha u(x) + (1-\alpha)u(y) \;\geq\; \min\{u(x), u(y)\},$$

so every concave function is quasiconcave. The converse fails because QC only requires the *weaker* "min" inequality (convex upper sets), which allows functions that curve *up* but have convex upper sets.

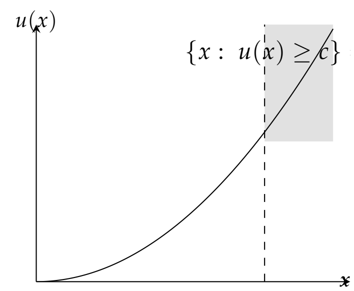


Figure 10: Quasiconcave but not concave: convex upper sets, graph curves upward.

Relation to (semi–)strict convexity of preferences. Semi–strict convexity of $\succeq$ corresponds to semi–strict quasiconcavity of some representative: if $x \succeq y$ and $x \neq y$, then mixing them strictly improves on y; in utility terms, $u(\alpha x + (1-\alpha)y) > u(y)$. Strict convexity of $\succeq$ corresponds to strict QC of (some) representing u, and—by invariance—of *every* strictly increasing transform of it.

Continuity

Definition (Continuity of preferences)**.** A preference relation $\succeq$ is *continuous* if for all $x \succ y$ there exists $\varepsilon > 0$ such that whenever $\|x' - x\| \leq \varepsilon$ and $\|y' - y\| \leq \varepsilon$, we still have $x' \succ y'$. Equivalently (on $X \subseteq \mathbb{R}^k$): all upper and lower contour sets are closed.[10]

[10] *Intuition for future me:* "Small perturbations don't flip strict rankings." No isolated jumps: ICs move smoothly (possibly with kinks).

Proposition (Continuous utility on $\mathbb{R}^k_+$ (Debreu))**.** *On $X = \mathbb{R}^k_+$, if a preference relation $\succeq$ is continuous $\iff$ there exists a* continuous *utility u that represents $\succeq$.*

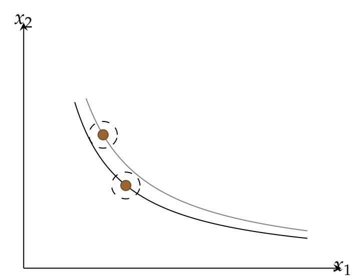


Figure 11: Continuity: small balls around $x \succ y$ keep the ranking.

Remark (What this *does* and *does not* say)**.** *If a continuous u represents $\succeq$, then $\succeq$ is continuous. If $\succeq$ is continuous, there exists a continuous representative u.*

However, it is **not** true that *every* utility representing a continuous $\succeq$ must be continuous. Any strictly increasing (possibly discontinuous) transform of a representative also represents $\succeq$.

Homotheticity

Definition (Homothetic preferences)**.** $\succeq$ is *homothetic* if for all $\lambda \geq 0$ and all x, y,

$$x \succeq y \implies \lambda x \succeq \lambda y.$$

"Scaling all goods by the same factor preserves rankings."

Example. Cobb–Douglas $u(x) = x_1^\alpha x_2^\beta$ is homogeneous of degree $\alpha + \beta$ and represents homothetic preferences. Perfect substitutes $u = a^\top x$ and perfect complements $u = \min_i\{a_i x_i\}$ are homogeneous of degree 1 (also homothetic).

Proposition (Homogeneous representation). *If a preference relation $\succeq$ is continuous and homothetic on $\mathbb{R}^k_+$, then it admits a representation by a* homogeneous *utility u of some degree $\rho > 0$, i.e. $u(\lambda x) = \lambda^\rho u(x)$ for all $\lambda \geq 0$. (By a monotone transform, one can normalize to degree 1.)*[11]

[11] *Intuition for future me:* Homothetic ⇒ indifference sets are radial blow-ups of one another; MRS depends on *proportions*, not on scale.

Quasi-linear preferences (a money good)

Definition (Quasi-linearity in good k). $\succeq$ is *quasi-linear in x_k* if there exists $v : \mathbb{R}^{k-1}_+ \to \mathbb{R}$ such that a representing utility can be written

$$u(x) = v(x_{-k}) + x_k, \qquad x = (x_{-k}, x_k).$$

Proposition (Characterization). *On $\mathbb{R}^k_+$, the following are equivalent:*[12]

[12] *Intuition for future me:* Quasi-linearity means "money" enters additively: adding the same c units of the numeraire to both bundles never changes the ranking. Consequence in demand: no income effects for the non-numeraire goods.

(a) *A preference relation $\succeq$ is continuous and quasi-linear in good k.*

(b) *(i)* Monotonicity in the numéraire (x_k): *for fixed x_{-k}, $x'_k > x_k \Rightarrow (x_{-k}, x'_k) \succ (x_{-k}, x_k)$.*
(ii) Translation invariance in the numeraire: *for all $c \in \mathbb{R}$,*

$$(x_{-k}, x_k) \succeq (y_{-k}, y_k) \iff (x_{-k}, x_k + c) \succeq (y_{-k}, y_k + c).$$

(c) Compensability via the numeraire: *for every $x_{-k}, y_{-k} \in \mathbb{R}^{k-1}_+$ there exist $x_k, y_k \in \mathbb{R}_+$ such that*

$$(x_{-k}, x_k) \sim (y_{-k}, y_k).$$

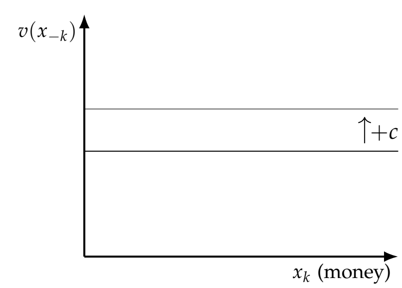


Figure 12: Quasi-linear: adding c to money shifts ICs vertically.

Intuition. Utility is additively separable and linear in the numeraire: money shifts utility one-for-one.

(a) With x_{-k} fixed, more money is strictly better.

(b) Adding the same c to money in both bundles preserves their ranking.

(c) Differences in non-numeraire goods can be offset by transferring money so the bundles can be made indifferent.

Appendix to Class 4: Curvature of u and induced preference convexity.

Property of u	**Upper contour sets**	**Preferences induced**	**Argmax on convex sets (demand)**
Strictly concave	Strictly convex	Strictly convex	Unique maximizer
Semi–strictly concave	Convex; strictness only when $u(x) \neq u(y)$	Convex (not necessarily strict)	Possibly multiple
Concave	Convex	Convex	Possibly multiple
Strictly quasiconcave	Strictly convex	Strictly convex	Unique maximizer
Semi–strictly quasiconcave	Convex; strict on chords across indifference	Strictly convex	Unique maximizer
Quasiconcave	Convex	Convex	Possibly multiple

Key facts. Concavity ⇒ QC. Strict versions (strictly concave / strictly QC / semi–strictly QC) ⇒ strictly convex preferences ⇒ uniqueness of the maximizer on convex feasible sets (e.g., budget sets). Non-strict versions yield convex (not strictly) preferences and allow multiplicity.

Outside the six cases (what if u is none of them?)

- **Continuous but not QC:** upper contour sets need not be convex ⇒ preferences are continuous but may be non-convex; maximizer sets on convex budgets can be disconnected or large; uniqueness not guaranteed.
- **Strictly increasing (monotone) but not continuous/QC:** preferences are monotone (more is better) but may fail continuity and convexity; existence/continuity of demand can fail without additional assumptions (e.g., closed/compact feasibility).
- **Arbitrary representatives (discontinuous transforms):** preserve ordinal rankings but can destroy regularity properties of the representative (continuity, differentiability) without changing the underlying (possibly regular) preferences.

Operational takeaway. For uniqueness and clean comparative statics, target **strict convexity of preferences** (achieved if u is strictly concave, strictly QC, or semi–strictly QC). With only convexity (concave / semi–strictly concave / QC), expect potential multiplicity.

Class 5: Separability and Choice vs. Demand

Additive separability (two blocks)

Separability asks when preferences can be decomposed into "blocks" of goods so that trade–offs within a block do not depend on the quantities of goods outside the block. In the simplest two–block *additively separable* case, overall utility is the sum of a subutility for each block; cross–block substitution disappears, and the interaction across blocks operates only through the budget.

$$u(x_1, \dots, x_\ell, x_{\ell+1}, \dots, x_k) = v(x_1, \dots, x_\ell) + \omega(x_{\ell+1}, \dots, x_k).$$

Definition (Weak separability into blocks $Y_1, \dots, Y_N$)**.** Let $Y_n \subseteq \{1, \dots, k\}$ be nonempty and pairwise disjoint ($Y_n \cap Y_{\tilde{n}} = \varnothing$ for $n \neq \tilde{n}$). Write $x_{Y_n} \in \mathbb{R}_+^{|Y_n|}$ for the subvector of x on coordinates in Y_n, and Y_n^c for the complement in $\{1, \dots, k\}$.

A preference relation $\succeq$ on $\mathbb{R}_+^k$ is *weakly separable into* $Y_1, \dots, Y_N$ if there exist subutility functions $V_n : \mathbb{R}_+^{|Y_n|} \to \mathbb{R}$ and an aggregator f that is increasing in each V_n such that

$$u(x) \;=\; f\big(V_1(x_{Y_1}),\, V_2(x_{Y_2}),\, \dots,\, V_N(x_{Y_N}),\, x_{(Y_1 \cup \cdots \cup Y_N)^c}\big),$$

and, equivalently, for every n, all $x_{Y_n}, \hat{x}_{Y_n} \in \mathbb{R}_+^{|Y_n|}$, and all common complements $x_{Y_n^c}, \hat{x}_{Y_n^c} \in \mathbb{R}_+^{|Y_n^c|}$ with $x_{Y_n^c} = \hat{x}_{Y_n^c}$,

$$(x_{Y_n}, x_{Y_n^c}) \succeq (\hat{x}_{Y_n}, x_{Y_n^c}) \iff V_n(x_{Y_n}) \geq V_n(\hat{x}_{Y_n}).$$

Acá hay que ver bien que se dice y poner intuición.

Definition (Strong separability (additive across blocks))**.** Let $Y_1, \dots, Y_N$ form a partition of $\{1, \dots, k\}$ (i.e., pairwise disjoint and $\bigcup_{n=1}^N Y_n = \{1, \dots, k\}$). A preference relation $\succeq$ is *strongly separable into* $Y_1, \dots, Y_N$ if it is weakly separable into this partition and admits the additive representation

$$u(x) \;=\; V_1(x_{Y_1}) + V_2(x_{Y_2}) + \cdots + V_N(x_{Y_N}).$$

Choice vs. Demand

Let $X \subseteq \mathbb{R}^k_+$ be the consumption set and let $\mathcal{E}$ denote the collection of all nonempty subsets of X.

Definition (Choice correspondence)**.** A *choice correspondence* is a map $C : \mathcal{E} \rightrightarrows X$ with $C(B) \subseteq B$ for every $B \in \mathcal{E}$.

Definition (Budget set)**.** For $p \in \mathbb{R}^k_{++}$ and $y > 0$,

$$B(p, y) \;=\; \{x \in \mathbb{R}^k_+ :\; p \cdot x \leq y\}.$$

Definition (Demand correspondence / function)**.** Given a preference relation $\succeq$ on X, the *demand correspondence* is

$$D(p, y) \;=\; \arg\max_{x \in B(p,y)} \;u(x) \quad \text{for any utility } u \text{ representing } \succeq .$$

When $D(p, y)$ is a singleton, we write $x(p, y)$ for its unique element and call it the *demand function*.

Consumer problem and basic properties

Definition (Consumer problem (CP))**.** Given $p \in \mathbb{R}^k_{++}$ and $y > 0$,

$$\max_{x \in \mathbb{R}^k_+} \;u(x) \qquad \text{s.t. } p \cdot x \leq y,$$

and $D(p, y)$ denotes the set of solutions.

Proposition. *Suppose the preference relation $\succeq$ is continuous (equivalently, admits a continuous utility u).*

(i) For every $p \in \mathbb{R}^k_{++}$ and $y > 0$, the CP has at least one solution: $D(p, y) \neq \varnothing$.

(ii) If $x \in D(p, y)$ and $\lambda > 0$, then $x \in D(\lambda p, \lambda y)$.

(iii) If u is quasi-concave (i.e., $\succeq$ is convex), then $D(p, y)$ is a convex set. If u is strictly quasi-concave, then $D(p, y)$ is a singleton.

(iv) If $\succeq$ is locally non-satiated, then every $x \in D(p, y)$ satisfies the budget: $p \cdot x = y$.

Sketch for (iii). Let $x, y \in D(p, y)$ and $\lambda \in [0, 1]$. *Feasibility:* the budget set is convex, so

$$p \cdot \big(\lambda x + (1 - \lambda) y\big) = \lambda \, p \cdot x + (1 - \lambda) \, p \cdot y \leq y,$$

hence $z := \lambda x + (1 - \lambda) y \in B(p, y)$. *Optimality under QC:* by quasi-concavity,

$$u(z) \;\geq\; \min\{u(x), u(y)\}.$$

Since x, y are optimal, $u(x) = u(y) = \max_{B(p,y)} u$, so $u(z) \geq \max_{B(p,y)} u$ and thus $z \in D(p, y)$. *Uniqueness under strict QC:* if $x \neq y$, strict quasi-concavity gives

$$u\big(\lambda x + (1-\lambda)y\big) \;>\; \min\{u(x), u(y)\} = \max_{B(p,y)} u,$$

a contradiction. Hence $D(p, y)$ is a singleton. $\square$

Appendix to class 5: Summary & Intuition

Weak separability (by blocks). Preferences are weakly separable into a partition $\{Y_1, \ldots, Y_N\}$ when the trade–offs *within* a block do not depend on quantities *outside* the block. Equivalently, there exist subutilities $V_n(x_{Y_n})$ and an aggregator f (increasing in each V_n) such that $u(x) = f(V_1, \ldots, V_N,\ x_{(Y_1 \cup \cdots \cup Y_N)^c})$. *Operational test (MRS-independence):* for $i, j \in Y_n$,

$$\frac{\partial}{\partial x_k}\left(\frac{u_i}{u_j}\right) = 0 \quad \forall\, k \notin Y_n,$$

so the within-block MRS is unaffected by goods outside Y_n. *Intuition:* choice can be done in stages—collapse each block to an index, then compare blocks via the budget.

Strong separability. Strong (additive) separability strengthens weak separability to

$$u(x) = \sum_{n=1}^{N} V_n(x_{Y_n}).$$

Quick checks: (i) If $u \in C^2$ and all cross-block second derivatives vanish, $\partial^2 u / \partial x_i \partial x_j = 0$ for $i \in Y_r$, $j \in Y_s$, $r \neq s$, then u is additively separable across blocks. (ii) If $u = f(V_1, \ldots, V_N)$ with a CES or product aggregator, a monotone transform (e.g. $\phi = \log$ or a power) makes $\phi \circ u$ additively separable across blocks.

Choice vs. demand (quick facts). Let $B(p, y) = \{x \geq 0 : p \cdot x \leq y\}$ and $D(p, y) = \arg\max_{x \in B(p,y)} u(x)$. Under continuity: existence $D(p, y) \neq \varnothing$. Scaling: if $x \in D(p, y)$ then $x \in D(\lambda p, \lambda y)$ (homogeneity of degree zero). Under (strict) quasi-concavity: $D(p, y)$ is convex (a singleton if strict). With local non-satiation: any $x \in D(p, y)$ exhausts the budget $p \cdot x = y$.

What to write in proofs. (1) State the partition and apply the MRS-independence test for weak separability. (2) For strong separability, either verify vanishing cross-block Hessian terms (additivity) or exhibit a suitable monotone transform that turns the index aggregator into a sum.

Class 6: Consumer Problem (CP)

Setup.

- Consumers are price takers.
- Choice: $x \in \mathbb{R}^k_+$.
- Standard conditions yield a well–behaved optimization problem.
- Sets we use below are compact (closed and bounded).

Definition (Budget set)**.** For $p \in \mathbb{R}^k_{++}$ and $y > 0$,

$$B(p, y) = \{x \in \mathbb{R}^k_+ : \ p \cdot x \leq y\}.$$

Definition (Demand correspondence and indirect utility)**.** For any $p \in \mathbb{R}^k_{++}$ and $y > 0$, let

$$D(p, y) = \arg\max_{x \in B(p,y)} u(x) \quad \text{and} \quad v(p, y) = u(x^*) \ \text{ for any } x^* \in D(p, y).$$

$D(p, y)$ is a (possibly set–valued) demand correspondence; $v(p, y)$ is the indirect utility.

Proposition (Basic properties of the CP)**.** *(i) If u is continuous, then at least one solution exists: $D(p, y) \neq \varnothing$.*

(ii) If u is quasi–concave, then the set of solutions $D(p, y)$ is convex.

(iii) If u is strictly quasi–concave, then there is either no solution or exactly one solution.

(iv) If preferences are locally non–satiated, then any solution x^* satisfies $p \cdot x^* = y$.

(v) If $x^* \in D(p, y)$ and $\lambda > 0$, then $x^* \in D(\lambda p, \lambda y)$.

Proposition (Homogeneity of degree zero)**.** *For all $\lambda > 0$,*

$$D(\lambda p, \lambda y) = D(p, y) \qquad \textit{and} \qquad v(\lambda p, \lambda y) = v(p, y).$$

Demand and indirect utility

For any $p \in \mathbb{R}^k_{++}$ and $y > 0$, let $D(p, y)$ be the set of all solutions to the consumer problem. If $D(p, y) \neq \varnothing$, define the *indirect utility*

$$v(p, y) = u(x^*) \quad \text{for any } x^* \in D(p, y).$$

Note that $D(p, y)$ is a (possibly set-valued) *demand correspondence* and $v(p, y)$ is the *indirect utility function*.

Proposition (Homogeneity of degree zero). *For every $\lambda > 0$,*

$$D(\lambda p, \lambda y) = D(p, y) \qquad \text{and} \qquad v(\lambda p, \lambda y) = v(p, y).$$

Intuition: *rescaling both prices and income by the same factor leaves the set of solutions unchanged.*

Proposition. *If u is continuous, then:*

(i) D is nonempty-valued;

(ii) D is upper hemicontinuous (the relevant continuity notion for correspondences);

(iii) v is continuous.

Theorem (Berge's Maximum Theorem). *If $f : T \times X \to \mathbb{R}$ is continuous and $\Gamma : T \rightrightarrows X$ is nonempty, compact-valued and continuous (upper and lower hemicontinuous), then the value function $v(t) = \max_{x \in \Gamma(t)} f(t, x)$ is continuous and the argmax $A(t) = \arg\max_{x \in \Gamma(t)} f(t, x)$ is nonempty, compact-valued, and upper hemicontinuous.*

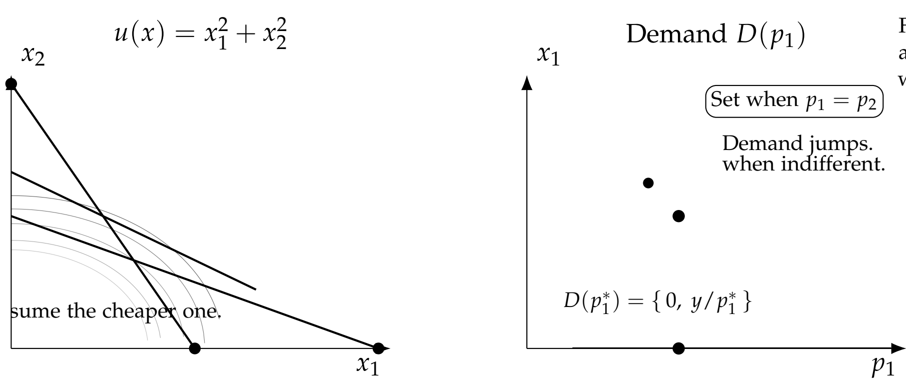


Figure 13: Left: convex utility ⇒ corner at the cheaper good. Right: non-QC case with a jump in $D(p_1)$.

Upper hemicontinuity of demand

Definition (Upper hemicontinuity (UHC)). A correspondence $F : \Theta \rightrightarrows X$ is *upper hemicontinuous at θ^** if for every sequence $\theta_n \to \theta^*$ and every sequence $x_n \in F(\theta_n)$ such that $x_n \to x^*$, we have $x^* \in F(\theta^*)$. We say F is UHC if this holds at every θ^*.

Definition (UHC of Marshallian demand)**.** Let $L \in \mathbb{N}$, $\Theta = \mathbb{R}^L_{++} \times \mathbb{R}_+$ with elements (p, y), and $X = \mathbb{R}^L_+$. The (Marshallian) demand correspondence is $D : \Theta \rightrightarrows X$. We say that D is *upper hemicontinuous at* (p^*, y^*) if for every sequence $(p^n, y^n) \to (p^*, y^*)$ and every sequence $x^n \in D(p^n, y^n)$ such that $x^n \to x^*$, we have $x^* \in D(p^*, y^*)$. We say D is UHC if this holds for every $(p^*, y^*) \in \Theta$.

Remark (Intuition)**.** As prices and income (p, y) vary slightly, track any optimal bundles $x_n \in D(p_n, y_n)$. If $x_n \to x^*$, upper hemicontinuity says x^* remains optimal at the limit (p^*, y^*). **Heuristic:** the set of optimal choices can *shrink* in the limit but cannot *sprout new points outside*; limit points of optimal selections never "jump out" of the limiting demand set.

Remark (How Kreps phrases it)**.** In Euclidean spaces Kreps uses the sequential version above (often called *upper semicontinuity*). Intuitively: limit points of selections cannot "jump outside" the limiting set; values may shrink but not expand.

Proposition (Closed–graph characterization)**.** *If F has nonempty compact values, then F is UHC $\iff$ its graph $\mathsf{Gr}(F) = \{(\theta, x) : x \in F(\theta)\}$ is closed.*

Proposition (Demand is UHC under continuity)**.** *If u is continuous, then for every $y > 0$ the demand correspondence $D(\cdot, y) : \mathbb{R}^k_{++} \rightrightarrows \mathbb{R}^k_+$ is nonempty, compact–valued, and UHC. (Proof idea: Berge's Maximum Theorem.)*

Intuition.

- *UHC as "no new limits":* if prices $p_n \to p^*$ and you pick any $x^n \in D(p_n, y)$, then every limit x^* is still optimal at (p^*, y). So optimizers cannot appear outside $D(p^*, y)$ in the limit.
- *Why only "upper"?* The set at the limit may be *smaller* (some options disappear), but cannot be *larger* than what limits of selections deliver.
- *Closed graph:* "limits of feasible pairs stay feasible" is exactly the closedness of $\mathsf{Gr}(D)$, hence the equivalence above.

Remark (Why UHC matters for equilibria)**.** Fixed–point theorems used for existence typically require correspondences that are nonempty,

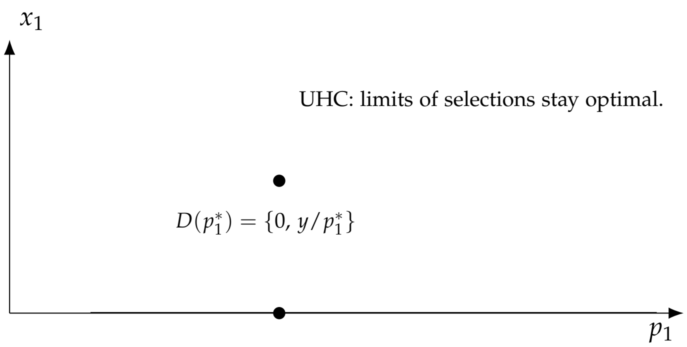


Figure 14: UHC in the "jump" case: if $p_n \to p_1^*$ and $x^n \in D(p_n, y)$ with $x^n \to x^*$, then $x^* \in D(p_1^*, y)$.

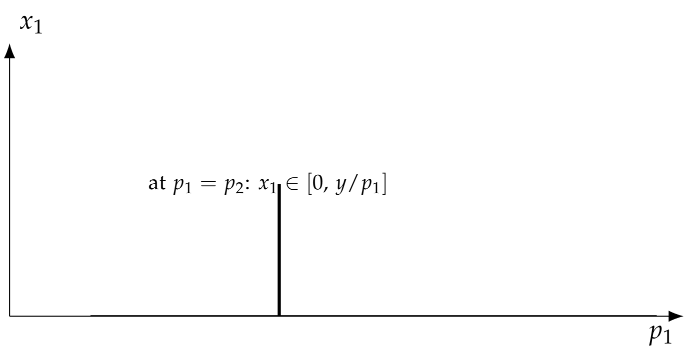


Figure 15: Straight indifference curves (perfect substitutes): similar picture but *without* the gap at $p_1 = p_2$.

compact–valued and UHC (e.g., Kakutani also needs convex–valued). If demand were not UHC, limit arguments in equilibrium proofs can fail; even with UHC alone, additional properties (e.g., convexity of values, aggregate feasibility) are needed to close the existence proof.

Differentiable case: KKT/FOC

Assume $u : \mathbb{R}^k_+ \to \mathbb{R}$ is differentiable. Consider

$$\max_{x \in \mathbb{R}^k_+} u(x) \quad \text{s.t.} \quad p \cdot x \le y, \quad x \ge 0.$$

Lagrangian:

$$\mathcal{L}(x, \lambda, \mu) \;=\; u(x) \;+\; \lambda\,(y - p \cdot x) \;+\; \mu \cdot x, \qquad \lambda \ge 0,\ \mu \ge 0.$$

Definition (First–order (KKT) conditions)**.** A feasible x^* satisfies the *FOC/KKT* if there exist multipliers $\lambda^* \ge 0$ and $\mu^* \ge 0$ such that

$$\begin{aligned}
\text{(stationarity)} \quad & \nabla u(x^*) - \lambda^* p - \mu^* = 0, \\
\text{(primal feas.)} \quad & p \cdot x^* \le y, \quad x^* \ge 0, \\
\text{(dual feas.)} \quad & \lambda^* \ge 0, \quad \mu^* \ge 0, \\
\text{(compl. slackness)} \quad & \lambda^*\,(y - p \cdot x^*) = 0, \qquad \mu_j^*\, x_j^* = 0 \;\; \forall j.
\end{aligned}$$

Equivalently, for each j,

$$\frac{\partial u}{\partial x_j}(x^*) \ \le \ \lambda^* p_j, \quad \text{with equality if } x_j^* > 0, \qquad \text{and} \qquad p \cdot x^* \le y, \text{ with equality if } \lambda^* > 0.$$

Proposition. *Two important points:*

(i) If x^* solves the consumer problem, then x^* satisfies the FOC/KKT.

(ii) If u is concave and the constraints are convex, then any x^* that satisfies the FOC/KKT is (globally) optimal. In particular, if there is a unique KKT solution, it is the unique global maximizer.

At the optimum (marginal utility per dollar). From the KKT,

$$\frac{1}{p_i}\frac{\partial u}{\partial x_i}(x^*) \ \le \ \lambda^* \quad \text{for all } i, \qquad \text{and if } x_i^* > 0: \ \frac{1}{p_i}\frac{\partial u}{\partial x_i}(x^*) = \lambda^*.$$

Hence, for any goods i, j that are consumed in positive amounts,

$$\frac{1}{p_i}\frac{\partial u}{\partial x_i}(x^*) = \frac{1}{p_j}\frac{\partial u}{\partial x_j}(x^*) = \lambda^* \quad \text{(“marginal utility in dollars” equalized).}$$

Envelope (income). If the indirect utility $v(p, y)$ is differentiable in y, then by the envelope theorem

$$\frac{\partial v(p, y)}{\partial y} = \lambda^*,$$

where λ^* is the multiplier at the optimum corresponding to (p, y).

Example: $u(x_1, x_2) = x_1 + \sqrt{x_2}$

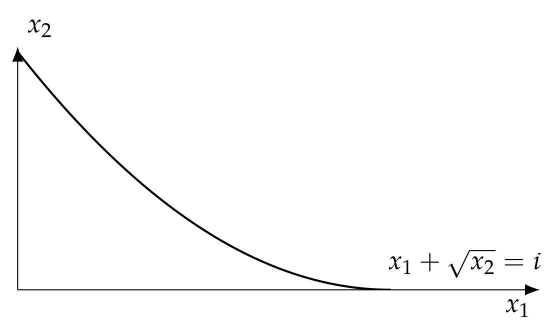


Figure 16: Indifference curve for $u(x_1, x_2) = x_1 + \sqrt{x_2}$ at level $\bar{u}$: $x_2 = (\bar{u} - x_1)^2$.

With prices $p = (p_1, p_2) \in \mathbb{R}^2_{++}$ and income $y > 0$, let $\lambda \ge 0$ be the multiplier on $p \cdot x \le y$ and $\mu \ge 0$ on $x \ge 0$. The KKT conditions are, for $j = 1, 2$,

$$\frac{\partial u}{\partial x_j}(x^*) \le \lambda p_j \quad (= \text{if } x_j^* > 0), \qquad p \cdot x^* \le y \quad (= \text{if } \lambda > 0).$$

Here

$$\frac{\partial u}{\partial x_1} = 1, \qquad \frac{\partial u}{\partial x_2} = \frac{1}{2\sqrt{x_2}} \quad (x_2 > 0).$$

Case 1 (interior: $x_1^* > 0$, $x_2^* > 0$). Equalities hold:

$$1 = \lambda p_1 \text{ (which implies } \lambda > 0) \qquad \frac{1}{2\sqrt{x_2^*}} = \lambda p_2 \;\Rightarrow\; \lambda = \frac{1}{p_1}, \quad x_2^* = \left(\frac{p_1}{2p_2}\right)^2.$$

Binding budget gives

$$x_1^* = \frac{y}{p_1} - \frac{p_1}{4p_2}.$$

Feasibility requires $x_1^* \geq 0$, i.e.

$$y \;\geq\; \frac{p_1^2}{4p_2}.$$

Case 2 (corner: $x_1^* = 0$, $x_2^* > 0$). Then $p_2 x_2^* = y \Rightarrow x_2^* = \frac{y}{p_2}$ and

$$1 \leq \lambda p_1, \qquad \frac{1}{2\sqrt{x_2^*}} = \lambda p_2 \;\Rightarrow\; \lambda = \frac{1}{2\sqrt{p_2 y}}, \quad 1 \leq \frac{p_1}{2\sqrt{p_2 y}} \;\Leftrightarrow\; y \;\leq\; \frac{p_1^2}{4p_2}.$$

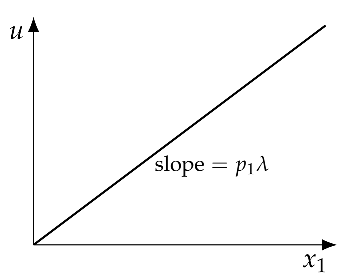
Demand correspondence.

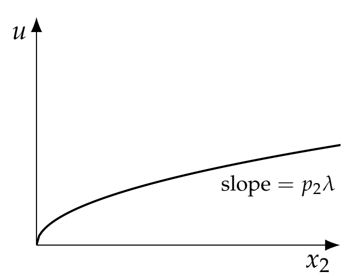
$$D(p_1, p_2, y) = \begin{cases} \left(\dfrac{y}{p_1} - \dfrac{p_1}{4p_2}\,,\; \left(\dfrac{p_1}{2p_2}\right)^2\right), & \text{if } y \geq \dfrac{p_1^2}{4p_2}, \\[2ex] \left(0\,,\; \dfrac{y}{p_2}\right), & \text{if } y \leq \dfrac{p_1^2}{4p_2}. \end{cases}$$

(At $y = \frac{p_1^2}{4p_2}$ both expressions coincide.)

"MU per dollar" equalization (interior). If $x_1^*, x_2^* > 0$ then

$$\frac{1}{p_1} = \frac{1}{p_2} \cdot \frac{1}{2\sqrt{x_2^*}} \quad \Rightarrow \quad x_2^* = \left(\frac{p_1}{2p_2}\right)^2.$$





Appendix to Class 6: Solving the Consumer Problem — A General Playbook

Standing notation. Prices $p \in \mathbb{R}^k_{++}$, income $y > 0$, choice $x \in \mathbb{R}^k_{+}$,

- Budget set $B(p, y) = \{x \geq 0 : \; p \cdot x \leq y\}$
- Demand correspondence $D(p, y) \subseteq \arg\max_{x \in B(p,y)} u(x)$
- Indirect utility $v(p, y) = \max_{x \in B(p,y)} u(x)$.

What graders expect to see (principles before math)

(a) **Meta-properties first.** State continuity / (strict) quasi-concavity of u and convexity/compactness of $B(p, y)$. This pins down existence, convexity of D, and (under strict qc) uniqueness, i.e. $D(p, y)$ is unique-valued (a marshallian demand function). If u is concave and constraints are convex, KKT are *necessary and sufficient*.

- *If u is continuous* and $B(p, y)$ is nonempty and compact, then by **Weierstrass** a maximizer *exists*: $D(p, y) \neq \emptyset$.
- *If u is quasi-concave*, then $D(p, y)$ is convex (possibly set-valued); *if u is strictly quasi-concave*, any maximizer is unique.
- *If u is concave (or strictly concave)* and constraints are convex, then any KKT point is a global maximizer (KKT are necessary *and* sufficient; uniqueness if strict concavity holds).
- *If u is not quasi-concave* (e.g., convex utility), KKT are *only necessary*: generate interior and corner candidates via KKT and then compare $u(\cdot)$ across candidates.

KKT quick implications by utility class (with optimality status).

- **Cobb–Douglas / CES (concave).** Interior solution via MU$/p$ equalization; then use $p \cdot x = y$ to get closed-form shares. *KKT status:* **necessary & sufficient** (concave objective, convex feasible set). *Uniqueness:* yes if strictly qc (e.g., all weights > 0; for CES in the concave range). *Corners:* only if parameters force a good to drop out (weight $= 0$) or feasibility binds at nonnegativity.
- **Quasi-linear (x_1 as numeraire).** $\lambda^* = 1/p_1$; solve the non-linear block with MU$/p = 1/p_1$ and allocate residual income to x_1. *KKT status:* **necessary & sufficient** if $v(\cdot)$ is concave. *Uniqueness:* yes if the non-linear block is strictly concave. *Corners:* arise when income is too small to hit the interior for the non-linear block.
- **Perfect substitutes ($u = \sum a_i x_i$).** Compare a_i/p_i; spend all income on any argmax. At ties, $D(p, y)$ is the full budget segment spanned

Why KKT are necessary & sufficient (intuition). In a concave maximization over a convex feasible set, the "surface" of u is bowl-shaped downward: any local improvement is also a global one. KKT encode two ideas: (i) *no profitable local move* (the gradient cannot point into the feasible set at the optimum, except along flat directions), and (ii) *which constraints pinch* (complementary slackness identifies binding constraints and zeros). Equalizing MU per dollar means a one-dollar reallocation cannot raise u; with concavity, "no local gain" ⇒ "no global gain". Conversely, any true maximizer must satisfy the same trade-offs and pinching, hence KKT are also necessary.

Necessary vs. sufficient (plain words). *Necessary:* every true optimum must satisfy KKT; if a point violates KKT, it cannot be optimal. *Sufficient:* any point that satisfies KKT is (globally) optimal. *Necessary & sufficient:* equivalence (KKT is both a filter and a certificate). If only necessary (non-concave u), KKT can produce *false positives* (e.g., saddles), so you **must compare** utilities across KKT-generated candidates.

Slater (CQ) for the Consumer Problem. *General:* For convex programs $\max u(x)$ s.t. $g_i(x) \leq 0$, Slater holds if there exists a point with all $g_i(x) < 0$ (strict feasibility). *Here:* $g_0(x) = p \cdot x - y \leq 0$ and $g_j(x) = -x_j \leq 0$. If $p \in \mathbb{R}^k_{++}$ and $y > 0$, pick $x = \varepsilon \mathbf{1} \gg 0$ with ε small: then $p \cdot x < y$ and $-x_j < 0$ for all j ⇒ Slater holds. *Implications:* With concave u and linear constraints, strong duality; KKT are *necessary & sufficient* (use subgradients if u has kinks). *Relation to MFQC:* MFQC is a smooth first-order CQ. In convex problems, Slater ⇒ MFQC and is easier to verify here. *Edge cases:* If $y = 0$ or some $p_i = 0$, strict feasibility may fail; handle separately (often a corner like $x = 0$).

by tied goods. *KKT status:* **necessary & sufficient** (linear $\equiv$ concave); may yield *many* KKT solutions at ties. *Uniqueness:* no at price-ratio ties (set-valued demand along the budget line). *Practical tie-handling:* report the convex set of budget-feasible mixtures of tied goods.

- **Perfect complements (Leontief, $u = \min\{x_i/\alpha_i\}$).** Fixed proportions at the kink; the budget pins the scalar multiple. *KKT status:* with *subgradients*, **necessary & sufficient** (concave as a minimum of linear forms). *Uniqueness:* yes (unique kink bundle scaled by y). *Corners:* non-issue—solution sits at the kink (not an "interior" in the smooth sense).
- **Convex utilities (non-qc; e.g., $u = x_1^2 + x_2^2$).** Interior FOCs do not characterize a maximum; solutions push to corners at the cheapest good(s). *KKT status:* **only necessary** (can flag stationary points that are *not* global maxima). *Workflow:* generate interior/corner candidates via KKT, then *compare* u and partition (p, y) by the verification inequalities. *Ties:* when price ratios equalize, demand is set-valued (entire budget segment).

Constraint qualification (Slater). In the consumer problem with $p \in \mathbb{R}^k_{++}$, $y > 0$, there exists $x \gg 0$ with $p \cdot x < y$; hence Slater holds. For concave u and convex constraints, KKT is *necessary and sufficient*. With kinks (nondifferentiable u), replace gradients by *subgradients* and the same logic applies.

(b) **Budget binds under monotonicity.** If preferences are locally non-satiated (e.g., monotone in each good), then any optimizer satisfies $p \cdot x^* = y$.

(c) **KKT written in "GSI format".** For each j,

$$\frac{\partial u}{\partial x_j}(x^*) \ \le\ \lambda p_j \quad (= \text{if } x_j^* > 0), \qquad p \cdot x^* \le y \quad (= \text{if } \lambda > 0).$$

This line is your workhorse; it drives all the case analysis.

(d) **Case-by-case structure.** Interior: equalize "MU per dollar" and use the binding budget to solve. Corners: set $x_k = 0$, solve for λ and remaining x, then *verify* the inequality on the zeroed good(s).

(e) **Partition the parameter space.** Convert the verification inequalities into clean thresholds in (p, y). Report $D(p, y)$ (and $v(p, y)$ if asked) *piece-wise*.

(f) **If u is not concave.** Say it explicitly. Treat KKT as *necessary* only, enumerate candidates (interior/corners), and compare $u(\cdot)$ across candidates.

(g) **One-line regularity checks.** Homogeneity of degree zero, Walras' law ($p \cdot x(p, y) = y$ when monotone), monotonicity in y, and (when relevant) UHC via Berge.

Cookie-cutter for Marshallian demand (exam version)

1. *State properties:* u continuous (and, if true, concave/strictly qc); $B(p,y)$ convex/-compact. *Therefore* existence (and uniqueness if strict qc) of D.
2. *Write KKT succinctly:*
$$\nabla u(x^*) - \lambda p - \mu = 0, \quad p \cdot x^* \le y,\ x^* \ge 0$$
Equivalently: $\partial u/\partial x_j \le \lambda p_j$ (= if $x_j^* > 0$) and budget binds if $\lambda > 0$.
3. *Show budget binds:* argue local non-satiation/monotonicity $\Rightarrow \lambda > 0$ and $p \cdot x^* = y$.
4. *Interior candidate:* if plausible, equalize MU per dollar $\frac{1}{p_i}\frac{\partial u}{\partial x_i} = \frac{1}{p_j}\frac{\partial u}{\partial x_j}$ for all i, j with $x_i^* > 0$, substitute into $p \cdot x = y$, solve x^*, check $x^* \ge 0$.
5. *Corner candidates:* choose subset S with $x_j = 0$ for $j \notin S$, solve on S with equalities, then *verify* $\partial u/\partial x_j \le \lambda p_j$ at $j \notin S$. This yields threshold conditions in (p, y).
6. *Partition and report:* translate verifications into clean regimes and give
$$D(p,y) = \begin{cases} x^{(1)}(p,y), & \text{if condition 1} \\ x^{(2)}(p,y), & \text{if condition 2} \\ \{x^{(2)}(p,y), x^{(3)}(p,y)\}, & \text{if ties} \end{cases}$$
7. Find $v(p,y) = u(x^{(\cdot)}(p,y))$.
8. *(If asked) regularity quick-checks:* homogeneity 0, Walras' law, monotonicity in y, UHC via Berge.

Exam-time diagnostics (fast checks)

- **Homothetic** $u \Rightarrow$ Engel curves are straight lines through the origin; $x(p,y) = y\,h(p)$ where $h(p)$ is a vector of optimal consumption when income is 1, $h(p) := x(p,1)$.
- **If a good is strictly essential** (e.g., complements): interior in that good unless y too small to reach the kink.
- **Strictly convex** u (non-qc) $\Rightarrow$ *corners* at the cheapest good; report ties as a set at price equalities.
- **MU/p equalization** is the quickest route for interiors; the binding budget closes the system.
- **Tie-handling:** if two regimes meet at a threshold, list *both* optimal bundles at the threshold (set-valued demand).

Playbooks for typical $u(\cdot)$

(i) Cobb–Douglas. $u(x) = \prod_{i=1}^k x_i^{\alpha_i}$, $\alpha_i > 0$.

Interior (always, with $x_i^* > 0$): $$\frac{\partial u/\partial x_i}{p_i} = \frac{\partial u/\partial x_j}{p_j} \Rightarrow x_i^* = \frac{\alpha_i}{\sum_\ell \alpha_\ell}\frac{y}{p_i}.$$

Properties:

- x_i is homogeneous of degree 0 in (p, y);
- expenditure shares= $\alpha_i / \sum \alpha_\ell$

(ii) CES with elasticity $\sigma > 0, \sigma \neq 1$. Let $u(x) = \left(\sum_{i=1}^{k} a_i\, x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$ with $a_i > 0$. Then the Marshallian demand is

$$x_i(p,y) \;=\; y\, \frac{a_i\, p_i^{-\sigma}}{\sum_{\ell=1}^{k} a_\ell\, p_\ell^{1-\sigma}}, \qquad \text{(and as } \sigma \to 1 \text{ this converges to Cobb–Douglas).}$$

Equivalent share form: $s_i = \frac{p_i x_i}{y} = a_i \left(\frac{p_i}{P}\right)^{1-\sigma}$ with $P = \left(\sum_\ell a_\ell p_\ell^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$.

(iii) Quasi-linear (numeraire x_1). $u(x) = x_1 + v(x_2, \ldots, x_k)$ with v concave, increasing.

$$\text{Rule:} \quad \frac{\partial v}{\partial x_j}(x^*) = \lambda p_j \;\; (j \geq 2), \quad \lambda = \frac{1}{p_1}.$$

Compute $(x_2^*, \ldots, x_k^*)$ from $\mathrm{MU}/p = 1/p_1$ and the budget for the non-linears; *residual* income goes to x_1:

$$x_1^* = \frac{y}{p_1} - \sum_{j \geq 2} \frac{p_j}{p_1} x_j^* \;\; \geq 0.$$

If y is too small to reach the interior for the non-linear block, you hit a corner there and the remainder goes to the numeraire.

(iv) Perfect substitutes. $u(x) = \sum_i a_i x_i$ with $a_i > 0$. Pick any good with highest a_i / p_i. If a unique maximizer i^* exists,

$$x_{i^*}^* = \frac{y}{p_{i^*}}, \quad x_j^* = 0 \; (j \neq i^*).$$

If ties at the top, $D(p, y)$ is the convex set of all bundles mixing tied goods on the budget line.

(v) Perfect complements (Leontief). $u(x) = \min\{x_1/\alpha_1, \ldots, x_k/\alpha_k\}$ with $\alpha_i > 0$. Fixed proportions at the kink:

$$x^* = \theta\,(\alpha_1, \ldots, \alpha_k), \quad \theta = \frac{y}{p \cdot \alpha} \quad \Rightarrow \quad x_i^* = \alpha_i \frac{y}{\sum_\ell p_\ell \alpha_\ell}.$$

(vi) Convex utilities (e.g., $u = x_1^2 + x_2^2$). Preferences are not quasi-concave; KKT are only necessary. Candidates are corners at the cheapest good:

$$x^* = \begin{cases} (y/p_1, 0), & p_1 < p_2, \\ (0, y/p_2), & p_2 < p_1, \\ \{(t, y/p_2 - (p_1/p_2)t) : \, t \in [0, y/p_1]\}, & p_1 = p_2. \end{cases}$$

Report demand piece-wise and be explicit about the set at $p_1 = p_2$.

(vii) "Power" or isoelastic blocks. For $u(x_1, x_2) = x_1^\beta + g(x_2)$ with $\beta \in (0,1)$, the interior condition is $\beta x_1^{\beta-1}/p_1 = g'(x_2)/p_2$ with budget binding. Proceed by solving one variable from MU/p and substituting into $p \cdot x = y$.

Berge's Maximum Theorem (consumer-specialized): existence, continuity, and demand regularity

Setup. Parameters $t := (p, y) \in \mathbb{R}^k_{++} \times \mathbb{R}_{++}$, feasible set $B(p, y) = \{x \in \mathbb{R}^k_+ : \ p \cdot x \leq y\}$, objective $u : \mathbb{R}^k_+ \to \mathbb{R}$ continuous. Define the demand correspondence $D(p, y) \subseteq \arg\max_{x \in B(p,y)} u(x)$ and indirect utility $v(p, y) := \max_{x \in B(p,y)} u(x)$.

Compactness at a glance. If $p \gg 0$ and $y > 0$, then $0 \leq x_i \leq y/p_i$ for each i, hence $B(p, y)$ is a nonempty, closed, bounded (thus compact) polytope.

Statement. If

(i) u is continuous,

(ii) for every (p, y) the feasible set $B(p, y)$ is nonempty and compact, and

(iii) the correspondence $(p, y) \mapsto B(p, y)$ is continuous (both upper and lower hemicontinuous),

Why (iii) holds here. With linear constraints and $p \gg 0$, small changes in (p, y) deform $B(p, y)$ continuously; no facets "appear/disappear" discontinuously. This delivers the continuity of the feasible-set mapping needed by Berge.

Then:

(a) **Existence & compact argmax:** $D(p, y)$ is nonempty and compact-valued.

(b) **Continuity of indirect utility:** $v(p, y)$ is continuous in (p, y).

(c) **Upper hemicontinuity (UHC) of demand:** $D(\cdot)$ is UHC in (p, y).

(d) **Convexity & uniqueness under curvature:** If u is (strictly) quasi-concave, $D(p, y)$ has convex (respectively, singleton) values; if u is strictly concave on a convex $B(p, y)$, D is a *continuous function*.

Class 7: Comparative statistics and GARP (Chapter 4)

Setup and notation

Let $D(p, y) \subseteq \mathbb{R}^L_+$ be the *demand correspondence* of a decision maker, represented by some preferences (or utility). For each price vector $p \in \mathbb{R}^L_+$ and income $y \in \mathbb{R}_+$, the set $D(p, y)$ contains the *optimal consumption bundles*. We write $x \in D(p, y)$ to indicate an optimal choice at (p, y).
[13]

[13] *Intuition for future me:* Think of $D(\cdot)$ as "what the consumer buys" at (p, y). We now ask what happens to demand when y, a good's own price p_i, or a cross-price p_j changes. At this stage we do not predict any specific relation, we just classify possibilities.

What can happen when y, p_i, or p_j changes

- **If income y changes:** good i can be
  - *Normal*: as y increases (with prices fixed), demand for i increases.
  - *Inferior*: as y increases (with prices fixed), demand for i decreases.
- **If a cross price p_j changes (with $j \neq i$):** good i can be
  - *Substitute*: as p_j increases, x_i increases (and decreases if p_j falls).
  - *Complement*: as p_j increases, x_i decreases (and increases if p_j falls).
- **If the own price p_i changes:** demand for i can be
  - *Normal*: the usual downward-sloping demand.
  - *Giffen*: demand for i increases when p_i increases.

Towards characterization

In the end, we want to characterize changes in p_i or p_j with two different effects. For example, we will record a change in prices and then describe the set of options that are weakly preferred.

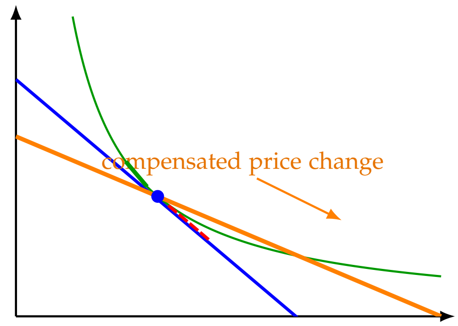


Figure 17: Compensated price change: the new budget line (orange) is tangent to the same indifference curve (green) at the optimum (blue point).

Compensated Law of Demand and a consequence for Giffen goods

Remark. (Budget exhaustion under LNS). If preferences are locally non–satiated (LNS) and $x \in D(p, y)$, then the budget is exhausted: $p \cdot x = y$. Same for any $x' \in D(p', y')$.

Proposition (Compensated law of demand (Slutsky inequality))**.** *Suppose preferences are LNS. Let $x \in D(p, y)$. Fix any $p' \in \mathbb{R}^L_+$ and define the* Slutsky compensation *$y' := p' \cdot x$. Let $x' \in D(p', y')$. Then*

$$(p' - p) \cdot (x' - x) \ \leq \ 0.$$

Compensated monotonicity. This proposition is exactly that compensated (Hicksian) demand is *decreasing* in prices. Hold utility fixed—stay on the same indifference curve—and ask which bundle is chosen. As prices tilt, the same-utility choice tilts the other way: less of what became relatively more expensive, more of what became relatively cheaper. This is a directional (not coordinatewise) monotonicity statement.

Proof. By budget exhaustion and the compensation $y' = p' \cdot x$, we have

$$p' \cdot x' \ = \ y' \ = \ p' \cdot x \quad \Rightarrow \quad p' \cdot (x' - x) = 0.$$

Since $x' \in D(p', y')$ and x is affordable at (p', y'), we have $x' \succeq x$. Because x solves the (p, y)-problem and $x' \succeq x$, local non-satiation implies $p \cdot x' \geq p \cdot x$. Hence

$$(p' - p) \cdot (x' - x) \ = \ p' \cdot (x' - x) - p \cdot (x' - x) \ = \ 0 - \big(p \cdot x' - p \cdot x\big) \ \leq \ 0.$$

□

Scalar vs. coordinatewise. When we write $p \cdot x' \geq p \cdot x$, this is a *single-number* (scalar) cost comparison at prices p—not a coordinatewise comparison of x' and x. Likewise, $(p' - p) \cdot (x' - x) \leq 0$ is an *inner product*: it aggregates price and quantity changes into one number that captures whether quantities move, on net, *against* the price change.

Remark. This result is about *compensated* price changes. It does not contradict the possibility of Giffen behavior, which is an *uncompensated* (Marshallian) phenomenon driven by the income effect.

Proposition (Giffen ⇒ Inferior)**.** *Suppose preferences are LNS. Fix (p, y) and let $x \in D(p, y)$. Consider a price change that raises only the own price of good i: $p'_i > p_i$ and $p'_j = p_j$ for $j \neq i$. Assume (Giffen behavior) that for the* uncompensated *problem at income y there exists $x^u \in D(p', y)$ with $x^u_i > x_i$. Define the compensated income $\bar{y} := p' \cdot x$ and pick $\bar{x} \in D(p', \bar{y})$. Then*

$$\bar{y} > y, \qquad \bar{x}_i \leq x_i < x^u_i,$$

so at prices p' a higher income $\bar{y}$ yields (weakly) less *of good i than a lower income y. Hence good i is (weakly)* inferior.

Proof. Since only p_i increases and $x_i \geq 0$, we have $\bar{y} = p' \cdot x > p \cdot x = y$. By the compensated law of demand applied to (p, y) and $(p', \bar{y})$ we get $(p' - p) \cdot (\bar{x} - x) \leq 0$, which here reduces to

$$(p'_i - p_i)\,(\bar{x}_i - x_i) \;\leq\; 0 \quad \Rightarrow \quad \bar{x}_i \leq x_i.$$

By the Giffen assumption for the *uncompensated* choice at (p', y), we have $x_i^u > x_i$. Putting these together,

$$\bar{x}_i \;\leq\; x_i \;<\; x_i^u,$$

so when income increases from y to $\bar{y}$ (holding p' fixed), demand for good i decreases. Thus i is inferior. □

Data set

We observe a finite set of choices:

$$\{(x_1, p_1, y_1),\, (x_2, p_2, y_2),\, \ldots,\, (x_n, p_n, y_n)\},$$

where each $x_i \in \mathbb{R}_+^L$ is the chosen bundle at prices $p_i \in \mathbb{R}_+^L$ and income $y_i = p_i \cdot x_i$ (by local non-satiation).

Revealed preference

Lemma. *Let a preference relation $\succeq$ be complete, transitive, and locally non-satiated preferences. If $x^* \in D(p, y)$ is chosen at prices p with income y, then:*

1. *For every x with $p \cdot x = y$, we have $x^* \succeq x$.*
2. *For every x with $p \cdot x < y$, we have $x^* \succ x$.*

Proof. (i) If $p \cdot x = y$, then x is feasible at (p, y). Since x^* is chosen, $x^* \succeq x$. (ii) If $p \cdot x < y$, local non-satiation implies there exists x' arbitrarily close to x with $x' \succ x$ and $p \cdot x' \leq y$. Then x' is feasible, so $x^* \succeq x'$. By transitivity, $x^* \succ x$. □

Definition (Direct revealed preference)**.** Given an observation (x_i, p_i, y_i) with $y_i \geq p_i \cdot x_i$:

- *Weak direct revealed preference:* if $p_i \cdot x_j \leq y_i$, then $x_i \succeq^d x_j$.
- *Strict direct revealed preference:* if $p_i \cdot x_j < y_i$, then $x_i \succ^d x_j$.

Definition (Revealed preference (transitive closure))**.** We write $x_i \succeq^R x_j$ if there exist indices $k_1, \ldots, k_m$ (possibly $m = 0$) such that

$$x_i \succeq^d x_{k_1}, \quad x_{k_1} \succeq^d x_{k_2},\ \ldots,\ x_{k_m} \succeq^d x_j.$$

If at least one step in the chain is strict, then $x_i \succ^R x_j$. In particular, allowing $m = 0$ gives $x_i \succeq^d x_j \Rightarrow x_i \succeq^R x_j$.

GARP

Definition (Generalized Axiom of Revealed Preference (GARP))**.** A data set satisfies GARP if there is no bundle x_i such that

$$x_i \succ^R x_i.$$

In words: revealed preference must not generate cycles with a strict step.

GARP intuition. Think "no improvement loops." (i) If a chosen bundle spends strictly less than income ($p^i \cdot x^i < y^i$), then it is *strictly* directly revealed preferred to itself ($x^i \succ^d x^i$), so GARP is immediately violated. (ii) More generally, if there is a chain of revealed weak preferences from x^i to x^j and another back from x^j to x^i, with at least one strict link, the two chains form a strict cycle ($x^i \succ^r x^i$). GARP = no such cycles.

Afriat's Theorem

Theorem (Afriat)**.** *The following statements are equivalent:*

1. *The data set can be rationalized by a complete, transitive, locally non-satiated preference relation.*
2. *The data set satisfies GARP.*
3. *The data set can be rationalized by a complete, transitive, locally non-satiated, convex, continuous, monotone preference relation.*

Class 8 – Choice under Uncertainty (Kreps Ch. 5)

We now move from deterministic choice to situations under uncertainty. The key idea is to represent actions by probability distributions over outcomes, and to assume that agents maximize expected utility.

Outcomes and Probability Distributions

Let X denote the set of outcomes. For simplicity, assume X is finite.

Definition (Probability Distributions over Outcomes)**.** We denote by $\Delta(X)$ the set of all probability distributions over X. Each $p \in \Delta(X)$ is a function $p : X \to [0, 1]$ with $\sum_{x \in X} p(x) = 1$.

Definition (Degenerate Distribution)**.** For each $x \in X$, let $\delta_x \in \Delta(X)$ be the degenerate (Dirac) distribution that assigns probability 1 to outcome x and 0 to all other outcomes. [14]

[14] *Intuition for future me:* We are identifying actions with distributions over outcomes. Deterministic choices are special cases represented by δ_x.

Preferences over Lotteries

Definition (Preference Relation over Lotteries)**.** Let $\succsim$ denote a preference relation defined on $\Delta(X)$. Formally, $\succsim \subseteq \Delta(X) \times \Delta(X)$. We write $p \succsim q$ if lottery p is weakly preferred to lottery q.

Properties of the Preference Relation

We impose axioms on $\succsim$ analogous to the deterministic setting:

(i) **Completeness and Transitivity.** The relation $\succsim$ is complete and transitive. [15]

[15] *Intuition for future me:* The agent can rank all lotteries consistently.

(ii) **Continuity.** Preferences are continuous: if $p \succ q \succ r$, then small perturbations of p or r preserve the ranking.

Under these two axioms, we already know that preferences can be represented by a utility function $U : \Delta(X) \to \mathbb{R}$ such that

$$p \succsim q \iff U(p) \geq U(q).$$

The Independence Axiom

We add a third crucial axiom:

Definition (Independence)**.** For all lotteries $p, q, r \in \Delta(X)$ and all $\alpha \in (0, 1)$,

$$p \succsim q \quad \Longrightarrow \quad \alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r.$$

This axiom captures the idea that preferences respect probabilistic mixtures.[16]

[16] *Intuition for future me:* If you prefer p to q, then you should also prefer any mixture of p with a third lottery r to the same mixture of q with r.

Representation by Expected Utility

Theorem (Von Neumann–Morgenstern Representation)**.** *If a preference relation $\succsim$ on $\Delta(X)$ satisfies completeness, transitivity, continuity, and independence, then there exists a utility function $u : X \to \mathbb{R}$ such that for all $p \in \Delta(X)$,*

$$U(p) = \sum_{x \in X} p(x)u(x),$$

and $p \succsim q \iff U(p) \geq U(q)$.[17] *Equivalently, there is $U : \Delta(X) \to \mathbb{R}$ given by $U(p) = \mathbb{E}_p[u] = \sum_x p(x)u(x)$ representing $\succsim$.*

$\Delta(X)$**: Representation vs. EU.** On the lottery space $\Delta(X)$, *completeness + transitivity + (topological) continuity* already guarantee the existence of some utility $U : \Delta(X) \to \mathbb{R}$ representing $\succsim$. *Independence is not needed for existence*; it is the axiom that forces U to be *affine in probabilities* (von Neumann–Morgenstern expected utility). With Independence (plus the usual Archimedean/mixture continuity), there is a cardinal index $u : X \to \mathbb{R}$ with $U(p) = \sum_{x \in X} p(x)\, u(x)$, unique up to positive affine transformations.

[17] *Intuition for future me:* Preferences over lotteries can be represented as the expected value of a utility index defined over outcomes.

Remark (Geometric picture and intuition)**.** On the 3–outcome simplex (coordinates $(p(x_1), p(x_2))$ with $p(x_3) = 1 - p(x_1) - p(x_2)$), expected utility is *affine* in probabilities. Hence: (i) indifference sets are *straight, parallel lines* with slope $-\frac{u(x_1)-u(x_3)}{u(x_2)-u(x_3)}$; (ii) each upper contour set $\{p : U(p) \geq \bar{U}\}$ is the intersection of a half–plane with the simplex (a convex "half–triangle"). Independence is exactly what yields this linearity in mixtures.

(a) Upper contour set in the simplex (b) Indifference lines: straight and parallel

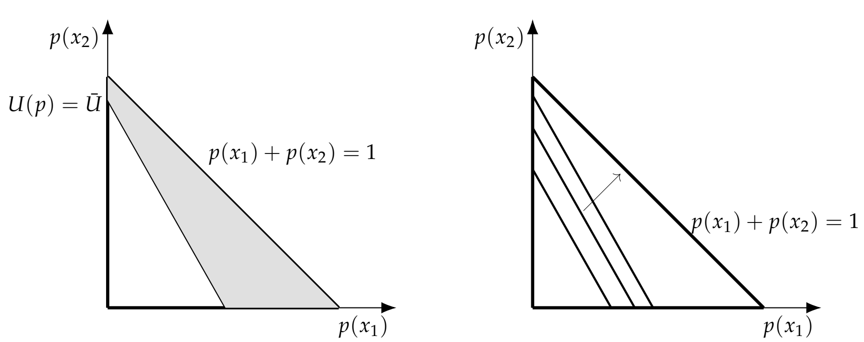


Figure 18: VNM on the 3–outcome simplex. Left: the upper contour set is the half–plane $U(p) \geq \bar{U}$ intersected with the simplex. Right: indifference sets are straight and parallel with slope $-(u_1 - u_3)/(u_2 - u_3)$.

Independence ⇒ Linearity and the Geometry of VNM

Proposition. *(Independence ⇒ Affine mixtures) Let X be finite and $\succsim$ on $\Delta(X)$ satisfy completeness, transitivity, continuity, and independence. The independence axiom implies the* affinity *(linearity in mixtures) of any utility representation.*

Lemma (Affinity). *If $U : \Delta(X) \to \mathbb{R}$ represents $\succsim$, then for all $p, q \in \Delta(X)$ and $\alpha \in [0,1]$,*

$$U\big(\alpha p + (1-\alpha)q\big) \;=\; \alpha\, U(p) \;+\; (1-\alpha)\, U(q).$$

Equivalently, writing each lottery componentwise,[18]

$$U\left(\sum_{x\in X} \big[\alpha p(x) + (1-\alpha) q(x)\big]\,\delta_x\right) = \alpha \sum_{x\in X} p(x) U(\delta_x) + (1-\alpha) \sum_{x\in X} q(x) U(\delta_x).$$

[18] *Intuition for future me:* Independence says that mixing with a common third lottery r preserves orderings. This forces U to be affine in probabilities, i.e., the utility of a mixture is the mixture of utilities.

Geometry on the simplex (three outcomes)

Let $X = \{x_1, x_2, x_3\}$ with $u_i := U(\delta_{x_i})$. Any $p \in \Delta(X)$ can be written as $p = (p_1, p_2, p_3)$ with $p_1 + p_2 + p_3 = 1$. By Lemma ,

$$U(p) = p_1 u_1 + p_2 u_2 + p_3 u_3 \;=\; (u_1 - u_3) p_1 + (u_2 - u_3) p_2 + u_3.$$

Hence each indifference set $\{p : U(p) = \bar{U}\}$ is a *straight line* inside the triangle $\{(p_1, p_2) \,:\, p_1 \ge 0,\ p_2 \ge 0,\ p_1 + p_2 \le 1\}$ given by

$$(u_1 - u_3) p_1 + (u_2 - u_3) p_2 \;=\; \bar{U} - u_3.$$

Upper contour sets $\{p : U(p) \ge \bar{U}\}$ are the corresponding half–triangles (convex regions).

Normalization and binary reduction

Because X is finite, there exist a *best* and a *worst* outcome w.r.t. $\succsim$: pick $\bar{x} \in X$ with $\delta_{\bar{x}} \succsim \delta_x$ for all x, and $\underline{x} \in X$ with $\delta_x \succsim \delta_{\underline{x}}$ for all x. Normalize the VNM index by[19]

$$u(\underline{x}) = 0, \qquad u(\bar{x}) = 1.$$

[19] *Intuition for future me:* Every lottery is *indifferent* to a two–point lottery between the best and worst outcomes, with weight on the best equal to its expected utility. Thus ranking lotteries is the same as ranking these binary mixtures by their weight on $\bar{x}$.

Then for any lottery $p \in \Delta(X)$,

$$U(p) = \sum_{x\in X} p(x)\, u(x) \;\in\; [0,1],$$

and, by affinity,

$$p \sim U(p)\, \delta_{\bar{x}} \;+\; \big(1 - U(p)\big)\, \delta_{\underline{x}}.$$

Remark. Writing $\alpha p + (1-\alpha) q$ componentwise gives

$$\sum_{x\in X} \big(\alpha p(x) + (1-\alpha) q(x)\big) u(x) = \alpha \sum_{x\in X} p(x) u(x) + (1-\alpha) \sum_{x\in X} q(x) u(x),$$

i.e., the indifference sets are level lines of a linear form in $(p(x))_{x\in X}$, which is exactly what the simplex plots above are visualizing.

Class 9 – Objective vs. Subjective Probabilities

In the theory of choice under uncertainty we can distinguish two versions:

(a) **Objective probabilities:** true probabilities for each outcome are given.

(b) **Subjective probabilities:** probabilities are not given; instead, from the choices of the agent we attempt to infer beliefs.

Objective probabilities and the VNM framework

Let X be a finite set of outcomes. Then $\Delta(X)$ denotes the set of all probability distributions over X, that is, all vectors $p = (p(x))_{x \in X}$ with $\sum_{x \in X} p(x) = 1$.

A preference relation $\succsim$ is defined over $\Delta(X)$, and we impose the following axioms:

(i) **Rationality (R).** $\succsim$ is complete and transitive.

(ii) **Continuity (C).** $\succsim$ is continuous, hence admits a continuous utility representation.

(iii) **Independence (I).** For all $p, q, r \in \Delta(X)$ and $\alpha \in (0, 1)$,

$$p \succsim q \implies \alpha p + (1-\alpha) r \succsim \alpha q + (1-\alpha) r.$$

Theorem (Von Neumann–Morgenstern Representation)**.** *If a preference relation $\succsim$ on $\Delta(X)$ satisfies (R), (C), and (I), then there exists a function $u : X \to \mathbb{R}$ such that for all $p, q \in \Delta(X)$,*

$$p \succsim q \iff \sum_{x \in X} p(x) u(x) \geq \sum_{x \in X} q(x) u(x).$$

Proof sketch. Part 1. Construct $U(p) = \sum_{x \in X} p(x) u(x)$ and show that this is well defined under (R), (C), (I).

Part 2. Show that $U(p)$ correctly orders the elements of $\Delta(X)$, i.e. $p \succsim q$ iff $U(p) \geq U(q)$. □

Remark (Strict separability)**.** The independence axiom induces a strong form of separability: the utility of a mixture is the corresponding mixture of utilities. This is exactly what makes expected utility a *linear* functional of probabilities.

Remark (Risk neutrality baseline)**.** The expected utility representation itself is agnostic about attitudes toward risk. Neutrality is the natural baseline. Risk aversion or risk seeking only appear once we apply u to monetary outcomes and study the curvature of u.

Rewriting lotteries

Given $p \in \Delta(X)$, we can always write it as a convex combination of degenerate lotteries:

$$p = \sum_{x \in X} p(x)\, \delta_x.$$

This allows us to rewrite rankings in terms of comparisons of degenerate lotteries weighted by probabilities. For instance, with three outcomes:

$$\begin{pmatrix} p(x) \\ p(y) \\ p(z) \end{pmatrix} = p(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + p(y) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + p(z) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

[20]

[20] *Intuition for future me:* The key is that rankings over lotteries should depend on whether probabilities differ. Expected utility treats these mixtures in a linear fashion.

Normalization and affine transformations

Suppose $\hat{u}(x) = \sum_{x \in X} p(x)\mu(x)$ represents $\succsim$. Then any other VNM index v that represents $\succsim$ must be an affine transformation:

$$v(x) = a\, \mu(x) + b, \quad a > 0,\, b \in \mathbb{R}.$$

Theorem (Uniqueness of VNM index)**.** *If u represents $\succsim$, then any other v representing $\succsim$ satisfies $v(x) = au(x) + b$ for all x, with $a > 0$.*

Remark (Expected utility as probability weights)**.** Expected utility assigns probabilities to outcomes in such a way that makes the agent indifferent between compound and reduced lotteries. This is the formal content of the independence axiom.

The Allais Paradox: when Independence fails

TB Revised

Example (Allais paradox)**.** Consider the following choices:

- Lottery A: $0.33 \to 27{,}000;\ 0.66 \to 24{,}000;\ 0.01 \to 0.$

- Lottery B: $1 \to 24{,}000$.

  Many agents prefer $B \succ A$.
  Now consider:

- Lottery A': $0.33 \to 27{,}000$; $0.67 \to 0$.

- Lottery B': $0.34 \to 24{,}000$; $0.66 \to 0$.

  Empirically, many agents prefer $A' \succ B'$.[21]

[21] *Intuition for future me:* But these two preference patterns are inconsistent under the independence axiom. The Allais paradox highlights that actual human choices systematically violate independence.

Remark. Replace $(\bar{M}, \bar{m}, 0)$ by (27,000, 24,000, 0) and $(0.10, 0.11, 0.89)$ by $(0.33, 0.34, 0.66)$. Your pairs become:

$$A : 0.33 \to 27{,}000,\ 0.66 \to 24{,}000,\ 0.01 \to 0, \qquad B : 1 \to 24{,}000,$$
$$A' : 0.33 \to 27{,}000,\ 0.67 \to 0, \qquad B' : 0.34 \to 24{,}000,\ 0.66 \to 0.$$

Then $B \succ A$ and $A' \succ B'$ is exactly the same violation: with $u(0) = 0$, $u(27{,}000) = 1$, $u(24{,}000) = x$, the first choice $B \succ A$ gives $x < \frac{33}{34}$, while the second $A' \succ B'$ gives $x > \frac{33}{34}$. Contradiction. Moreover,

$$\boxed{B \;=\; 0.66\,\delta_{24{,}000} \;+\; 0.34\,B', \qquad A \;=\; 0.66\,\delta_{24{,}000} \;+\; 0.34\,A'.}$$

Thus A and B are obtained by adding the *same* common consequence 0.66 of 24,000 to A' and B'. Independence would require preserving the A' vs. B' ordering, which is violated empirically.

Class 10 – Subjective Probabilities

Motivation

We aim to infer *subjective beliefs* from choices. The key shift in this part is that the *objects of choice are acts*—state–contingent plans—not bare outcomes or exogenously risky lotteries. If you repeatedly prefer the act "\$100 if A, \$0 otherwise" to the act "\$100 if B, \$0 otherwise," holding consequences fixed, your choices reveal that you *regard event A as more likely than B*. Preferences over acts thus encode both *tastes* over consequences and *beliefs* about states.

Intuition: An act is an *if–then* rule. We keep the *consequence menu* fixed and vary the *event* that triggers it. The way you rank such rules reveals a likelihood ordering over events.

Roadmap. (1) Define *states*, *events*, *consequences*, and *acts* (functions from states to consequences). (2) Take preferences over acts as primitive. (3) Impose behavioral axioms (including the Sure–Thing Principle) under which preferences admit a representation as *expected utility* with a *subjective probability* on events.

Framework

Definition (States, events, acts, and consequences)**.** Let S be a (finite) set of *states of the world* and let $\mathcal{E} \subseteq 2^S$ be the set of *events*. Let X be a set of *consequences*. An *act* is a function $a : S \to X$ that assigns a consequence to each state. A (complete and transitive) preference relation $\succeq$ is defined over the set of acts $\mathcal{A} = X^S$.

Remark (Why acts are the primitives here)**.** Unlike in objective–risk problems (where lotteries are given with known probabilities), here probabilities are *not* assumed; they are inferred from $\succeq$ over acts. Acts are the natural primitives because they tell us, for each possible state, *what would happen if this plan were chosen*. Events matter only through how acts condition consequences on them.

Remark (When consequences are monetary)**.** In many applications we take $X = [a, b] \subset \mathbb{R}$ (money or consumption) and we already endowed X with a vNM utility index $u : X \to \mathbb{R}$. We also write $\Delta(X)$ for the set of *simple lotteries* over X (probability distributions with finite support).

Remark (Important: timing)**.** We must choose an act $a \in \mathcal{A}$ *before* learning which state $s \in S$ will prevail. After the state realizes, the chosen act mechanically yields its corresponding consequence $a(s)$.

Representation: subjective expected utility (SEU)

The idea is that preferences over acts can be represented as expected utility with respect to a *subjective* probability on states.

Theorem (SEU representation (finite S; Anscombe–Aumann flavor))**.** *Suppose $\succeq$ over $\mathcal{A}$ satisfies the usual list of axioms:*[22] *then there exist (i) a probability distribution p on S and (ii) a utility index $u : X \to \mathbb{R}$ such that for all acts $a, b \in \mathcal{A}$,*

$$a \succeq b \quad \Longleftrightarrow \quad \sum_{s \in S} p(s)\, u\big(a(s)\big) \;\geq\; \sum_{s \in S} p(s)\, u\big(b(s)\big).$$

Moreover, p is unique, and u is unique up to a positive affine transformation $u'(x) = \alpha u(x) + \beta$ with $\alpha > 0$.

[22] Weak order, continuity, monotonicity w.r.t. consequences, and the Sure-Thing/independence axiom across events; plus nondegeneracy. In the AA setup, mixture independence is imposed via objective mixtures of consequences.

Reading: "Preferences reveal your subjective probabilities, and they admit an expected-utility representation."

Remark (Why the Sure-Thing Axiom matters)**.** The Sure-Thing Principle (STP) says: if two acts deliver the *same* consequence on an event E, then your ranking between them should depend only on what they do on E^c. In SEU, this follows from linearity of expectation and is **equivalent to independence** across events.

Add excercise 3 of PS5b as an example.

Ellsberg's paradox: a failure of STP (ambiguity)

Consider an urn with 90 balls: 30 are red (R); the remaining 60 are blue (B) or green (G) in unknown proportions (ambiguity about B vs. G). The state space is $S = \{R, B, G\}$. Pay \$100 if the described event occurs, and \$0 otherwise.

Act	R	B	G
A	100	0	0
B	0	100	0
A'	100	0	100
B'	0	100	100

Example (Typical choices)**.** Empirically, many subjects choose $A \succ B$ (bet on red) and also $B' \succ A'$ (bet on blue-or-green rather than red-or-green).

Remark (Why this contradicts SEU / STP)**.** Observe that A' and B' are obtained from A and B by adding the *same* payoff on event G (both give \$100 on G). By the Sure-Thing Principle, adding a common consequence on G should not flip the ranking: from $A \succ B$ we should infer $A' \succ B'$. The observed reversal $B' \succ A'$ therefore violates STP (and hence SEU). This is interpreted as *ambiguity aversion*: people dislike bets whose probabilities are ill-defined (here, the split between B and G).

Connection to "probabilistic sophistication": if preferences depended only on the induced lotteries over consequences, a single subjective p would rationalize both choices. Ellsberg shows they often do not.

Event comparisons via bets

Fix consequences and vary only the event that triggers the prize. For example, compare the two acts

$$a_E(s) = \begin{cases} x_H & \text{if } s \in E, \\ x_L & \text{if } s \notin E, \end{cases} \qquad a_F(s) = \begin{cases} x_H & \text{if } s \in F, \\ x_L & \text{if } s \notin F, \end{cases}$$

with $x_H \succ x_L$. If $a_E \succeq a_F$ for all such pairs (x_H, x_L), we read this as "E is (subjectively) at least as likely as F," which SEU rationalizes via $p(E) \geq p(F)$.

Utility for money and simple lotteries (Chapter 6)

We now specialize to monetary consequences. Let $X = [a, b] \subset \mathbb{R}$. Denote by $\Delta(X)$ the set of *simple lotteries* on X, i.e. probability distributions with finite support.

Definition (vNM index on money)**.** A preference $\succeq$ on $\Delta(X)$ has a vNM representation if there exists $u : X \to \mathbb{R}$ such that for all $\pi, \rho \in \Delta(X)$,

$$\pi \succeq \rho \quad \Longleftrightarrow \quad \sum_{x \in \text{supp}(\pi)} \pi(x)\, u(x) \;\geq\; \sum_{x \in \text{supp}(\rho)} \rho(x)\, u(x).$$

The function u is unique up to positive affine transformations.

Definition (Monotonicity on money)**.** Write $\delta_x \in \Delta(X)$ for the degenerate lottery that pays x for sure. A preference $\succeq$ is *increasing* if for all $x, y \in [a, b]$, $x \geq y$ implies $\delta_x \succeq \delta_y$ (strictly if $x > y$).

Remark. Intuition: "More money is (weakly) better for sure outcomes." In a vNM representation, this is equivalent to u being (weakly) increasing on $[a, b]$.

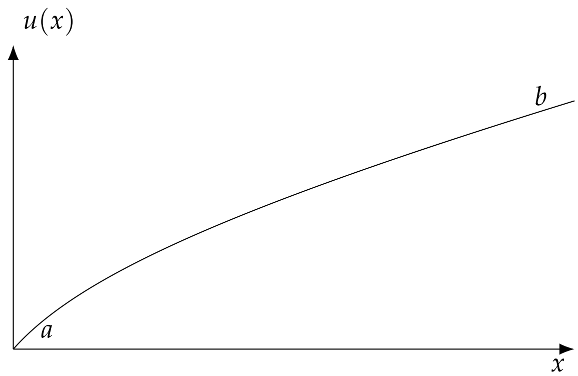


Figure 19: An increasing vNM index u on money.

First-order stochastic dominance (FOSD)

Definition (FOSD via CDFs)**.** For $\pi, \rho \in \Delta(X)$ with cumulative distribution functions F_π and F_ρ, we say that π *first-order stochastically dominates* ρ (write $\pi \succeq_{\text{FOSD}} \rho$) if

$$F_\pi(x) \;\le\; F_\rho(x) \quad \text{for all } x \in [a, b],$$

with strict inequality for some x (for strict FOSD). If the CDFs cross, the pair is not comparable by FOSD.

Proposition (Monotonicity $\iff$ FOSD)**.** *Let $\succeq$ on $\Delta(X)$ admit a vNM representation by u. Then*

$$\succeq \text{ is increasing } \iff u \text{ is (weakly) increasing } \iff \big(\forall\, \pi, \rho \in \Delta(X)\big)\ \pi \succeq_{\text{FOSD}} \rho \;\Rightarrow\; \pi \succeq \rho.$$

Proof sketch. *($\Rightarrow$) If $\delta_x \succeq \delta_y$ for $x \ge y$, then $u(x) \ge u(y)$, so u is increasing; FOSD implies higher probability on higher payoffs, and linearity of EU with increasing u gives $\pi \succeq \rho$. ($\Leftarrow$) If u were not increasing, there exist $x > y$ with $u(x) < u(y)$: but $\delta_x \succeq_{FOSD} \delta_y$ while EU would rank $\delta_y \succ \delta_x$, a contradiction.*

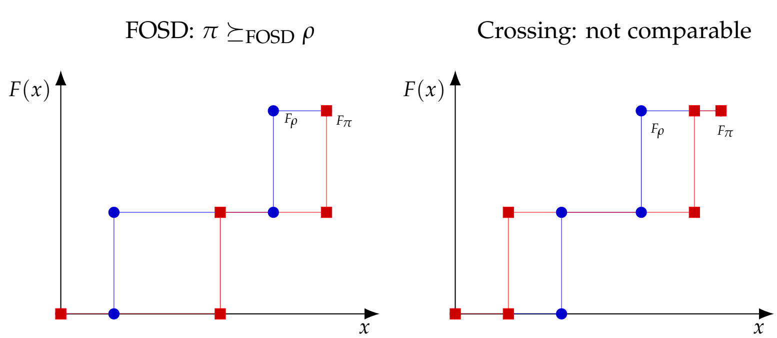


Figure 20: Left: $F_\pi \le F_\rho$ pointwise (FOSD). Right: CDFs cross $\Rightarrow$ no FOSD ranking.

Reading tip. If $F_\pi \le F_\rho$, every increasing u yields $\mathbb{E}_\pi[u(X)] \ge \mathbb{E}_\rho[u(X)]$. If CDFs cross, some increasing u prefers π and another prefers ρ.

Class 11 – Utility of Money (Chapter 6)

Setup and simple lotteries

Let $X = [0, \bar{x}] \subset \mathbb{R}$ denote monetary outcomes, and let $\Delta(X)$ be the set of *simple lotteries* (finite support probability distributions) over X. Preferences $\succeq$ are defined over $\Delta(X)$.

Definition (EU representation on money)**.** A vNM index $u : X \to \mathbb{R}$ represents $\succeq$ if for all $\pi, \rho \in \Delta(X)$,

$$\pi \succeq \rho \iff \underbrace{U(\pi)}_{:=\mathbb{E}_\pi[u(X)]} = \sum_{x \in \operatorname{supp}(\pi)} \pi(x)\, u(x) \;\geq \sum_{x \in \operatorname{supp}(\rho)} \rho(x)\, u(x) = U(\rho),$$

with u unique up to positive affine transformations.

Monotonicity and FOSD

Definition (Monotonicity on sure amounts)**.** Let δ_x be the degenerate lottery that pays x for sure. We say that $\succeq$ is *increasing* if for all $x \geq y$ in X, $\delta_x \succeq \delta_y$ (strictly increasing if $x > y \Rightarrow \delta_x \succ \delta_y$).

Proposition (Three equivalent statements)**.** *For preferences $\succeq$ on $\Delta(X)$ with EU index u, the following are equivalent:*

(i) $\succeq$ *is (weakly) increasing.*

(ii) *u is (weakly) increasing on $[0, \bar{x}]$.*

(iii) *If π first-order stochastically dominates ρ (i.e. $F_\pi(x) \leq F_\rho(x)$ for all x), then $\pi \succeq \rho$.*

Moreover, if $\succeq$ is strictly *increasing, then u is strictly increasing and strict FOSD implies strict preference ($F_\pi \leq F_\rho$ and $F_\pi < F_\rho$ somewhere $\Rightarrow \pi \succ \rho$).*

Risk aversion and SOSD

Definition (Risk aversion)**.** $\succeq$ is *risk-averse* if for every lottery $\pi \in \Delta(X)$,

$$\delta_{\mathbb{E}_\pi[X]} \succeq \pi.$$

(Strict risk aversion: the inequality is strict whenever π is non-degenerate.)

Proposition (Characterizations of risk aversion). *For EU preferences with index u, the following are equivalent:*

(a) $\succeq$ *is (strictly) risk-averse.*

(b) u *is (strictly) concave on* $[0, \bar{x}]$.

(c) (SOSD consistency) If π *second-order stochastically dominates* ρ *(write* $\pi \succeq_{SOSD} \rho$*), then* $\pi \succeq \rho$*. With strict concavity,* $\pi \succ \rho$ *when the dominance is strict.*

Sketch. *(a)⇔(b) is Jensen:* $u(\mathbb{E}[X]) \geq \mathbb{E}[u(X)]$*, with* $>$ *if* u *is strictly concave and* π *non-degenerate. (b)⇒(c): concavity plus the definition of SOSD (below) implies* $\mathbb{E}_\pi[u(X)] \geq \mathbb{E}_\rho[u(X)]$*. (c)⇒(a) by taking* $\rho = \pi$ *and* $\pi = \delta_{\mathbb{E}[X]}$*, which SOSD-dominates* π*.* □

Definition (Second-order stochastic dominance). Let $X = [0, \bar{x}]$. For $\pi, \rho \in \Delta(X)$ with CDFs F_π, F_ρ, we write

$$\pi \succeq_{\text{SOSD}} \rho \quad \Longleftrightarrow \quad \int_0^t F_\pi(x)\,dx \;\leq\; \int_0^t F_\rho(x)\,dx \;\text{ for all } t \in [0, \bar{x}],$$

and

$$\int_0^{\bar{x}} F_\pi(x)\,dx \;=\; \int_0^{\bar{x}} F_\rho(x)\,dx \quad \text{(equivalently, } \mathbb{E}_\pi[X] = \mathbb{E}_\rho[X]\text{)}.$$

Remark (Mean-preserving spread (MPS)). ρ is an MPS of π iff there exists a zero-mean *noise* $\tilde{Y}$ such that

$$X_\rho \;=\; X_\pi + \tilde{Y}, \qquad \mathbb{E}[\tilde{Y} \mid X_\pi] = 0.$$

This is equivalent to $\pi \succeq_{\text{SOSD}} \rho$. Intuition: ρ is obtained by "spreading" mass of π away from its mean without changing the mean.

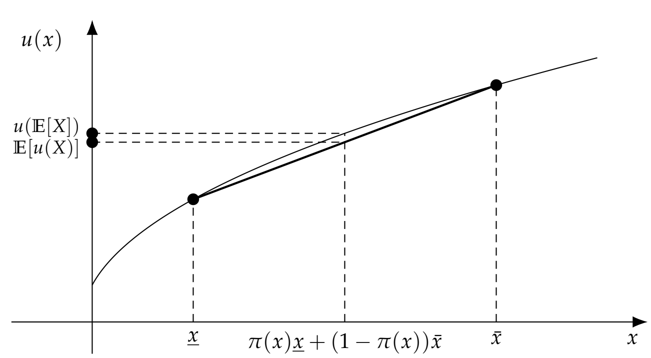


Figure 21: Risk aversion: concavity ⇒ Jensen $u(\mathbb{E}[X]) \geq \mathbb{E}[u(X)]$. Add a plot with risk loving.

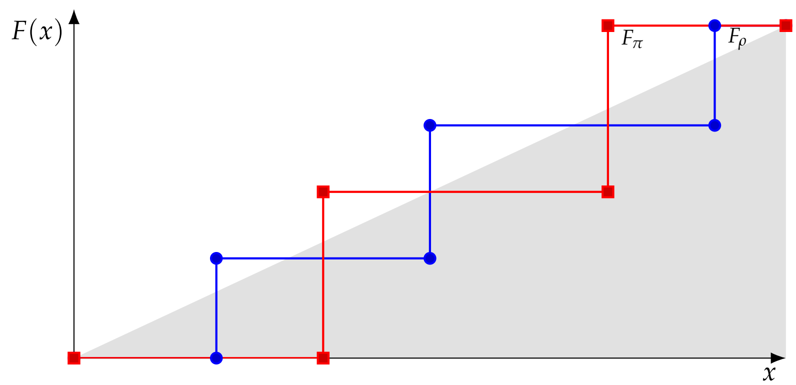


Figure 22: Second-order dominance: $\int_0^t F_\pi \le \int_0^t F_\rho$ for all t (same mean). Then every increasing concave u prefers π to ρ.

Takeaways

- Monotonicity on sure amounts $\iff$ u increasing $\iff$ FOSD consistency (Prop.).
- Risk aversion $\iff$ u concave $\iff$ SOSD consistency (Prop.).
- Mean-preserving spreads (zero-mean noise added) worsen lotteries for a risk-averse DM; they leave the mean unchanged but raise dispersion.

Arrow–Pratt, certainty equivalent, and risk premium

Definition (Arrow–Pratt absolute risk aversion)**.** For an EU index $u : X \to \mathbb{R}$ with $u' > 0$, the *absolute risk-aversion* (ARA) coefficient is

$$A_u(x) \;:=\; -\frac{u''(x)}{u'(x)}.$$

It is *invariant to positive affine transformations*: if $v = \alpha u + \beta$ with $\alpha > 0$, then $A_v \equiv A_u$.

Definition (Certainty equivalent and risk premium)**.** Given a lottery $\pi \in \Delta(X)$, the *certainty equivalent* $\mathrm{CE}(\pi) \in X$ is the (unique) number such that

$$u\big(\mathrm{CE}(\pi)\big) \;=\; \mathbb{E}_\pi\big[u(X)\big].$$

The *risk premium* is

$$\mathrm{RP}(\pi) \;:=\; \mathbb{E}_\pi[X] \;-\; \mathrm{CE}(\pi).$$

Proposition (Risk aversion $\Leftrightarrow$ nonnegative premia)**.** *For EU preferences with u increasing, the following are equivalent:*

(i) The DM is (weakly) risk-averse.

(ii) u is (weakly) concave.

(iii) *For every π, $\mathrm{RP}(\pi) \geq 0$ (strict $>$ for nondegenerate π when u is strictly concave).*

Sketch. *Jensen: $u(\mathbb{E}[X]) \geq \mathbb{E}[u(X)]$ with $>$ for strict concavity and nondegeneracy.* □

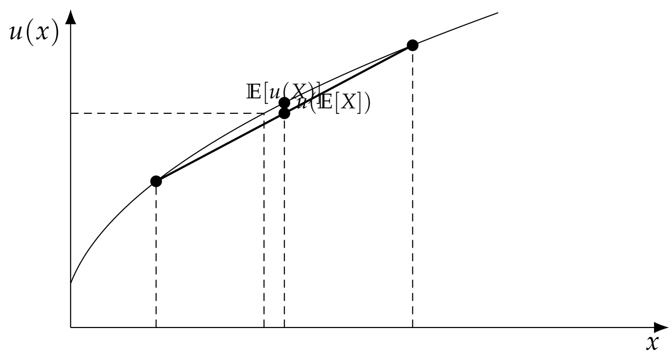


Figure 23: Certainty equivalent: $u(\mathrm{CE}) = \mathbb{E}[u(X)] \leq u(\mathbb{E}[X])$. Risk premium $= \mathbb{E}[X] - \mathrm{CE}$.

Small-risk expansion and Arrow–Pratt

Fix $x \in X$ and let $\tilde{Z}$ be a mean-zero r.v. with $\mathbb{E}[\tilde{Z}] = 0$ and $\mathbb{E}[\tilde{Z}^2] < \infty$. For $\varepsilon \geq 0$, define the risk premia $c(\varepsilon)$ by the *CE equation*

$$u\big(x - c(\varepsilon)\big) \;=\; \mathbb{E}\big[u\big(x + \varepsilon \tilde{Z}\big)\big].$$

Then $c(0) = 0$, $c'(0) = 0$, and if $u \in C^2$ and c is twice differentiable at 0,

$$\boxed{c''(0) \;=\; A_u(x)\, \mathbb{E}[\tilde{Z}^2] \;=\; -\frac{u''(x)}{u'(x)}\, \mathbb{E}[\tilde{Z}^2]}$$

so that, for small risks,

$$\mathrm{RP}\big(\varepsilon \tilde{Z}\big) \;=\; c(\varepsilon) \;\approx\; \tfrac{1}{2}\, A_u(x)\, \varepsilon^2\, \mathrm{Var}(\tilde{Z}).$$

Derivation. Apply a second-order Taylor expansion to both sides around $\varepsilon = 0$ and use $\mathbb{E}[\tilde{Z}] = 0$.

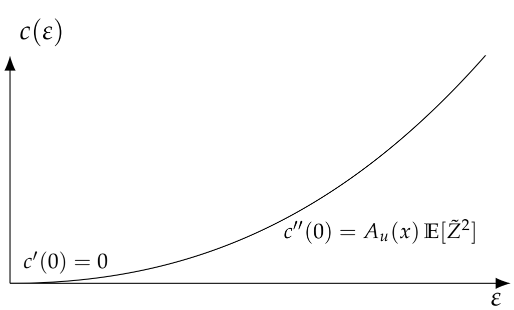


Figure 24: Small-risk premium $c(\varepsilon)$: tangent flat at 0 and convex when $A_u(x) > 0$.

Constant absolute risk aversion (CARA)

Definition (CARA)**.** u has *constant* ARA equal to $\rho > 0$ if $A_u(x) \equiv \rho$ for all x, i.e.

$$-\frac{u''(x)}{u'(x)} = \rho \quad \Longleftrightarrow \quad u''(x) + \rho\, u'(x) = 0.$$

Solving the ODE yields $u(x) = \alpha - \beta e^{-\rho x}$ with $\beta > 0$. A convenient normalization is $u(x) = 1 - e^{-\rho x}$.

Proposition (Comparative risk aversion (Pratt))**.** *If $A_u(x) \geq A_v(x)$ for all x, then u is uniformly more risk-averse than v (equivalently, $v = \phi \circ u$ for some increasing concave ϕ). Hence $\mathrm{RP}_u(\pi) \geq \mathrm{RP}_v(\pi)$ for all π.*

A simple portfolio choice "at zero"

Let initial wealth be $m > 0$. Choose a share $\lambda \in [0, 1]$ to invest in a risky increment $\tilde{X}$ while the remainder stays safe. Final wealth:

$$W(\lambda) = (1 - \lambda)\, m + \lambda\, \tilde{X}.$$

The DM chooses λ to maximize $E[u(W(\lambda))]$.

Proposition (Sign of the optimal direction at $\lambda = 0$)**.** *Suppose u is increasing and differentiable. Then*

$$\frac{d}{d\lambda}\, \mathbb{E}\big[u(W(\lambda))\big]\Big|_{\lambda=0} = u'(m)\, \big(\mathbb{E}[\tilde{X}] - m\big).$$

Hence, if an optimum is interior near 0, we have $\lambda^* > 0$ iff $\mathbb{E}[\tilde{X}] > m$. If $\tilde{X}$ is a *net* risky payoff over the safe (so the safe is 0), this simplifies to $\lambda^* > 0$ iff $\mathbb{E}[\tilde{X}] > 0$.

Remark. For (small) mean-zero risks with variance σ^2, the second-order effect is governed by Arrow–Pratt:

$$\frac{d^2}{d\lambda^2}\, \mathbb{E}\big[u(W(\lambda))\big]\Big|_{\lambda=0} = u'(m)\, A_u(m)\, \sigma^2 \ (-\text{sign}),$$

so risk aversion ($A_u > 0$) penalizes variance and pushes the optimum λ^* downward.

Class 12 – Theory of the Firm (Chapter 9)

Setup

We model the firm as a single decision maker. The *technology* is a production set

$$\mathcal{Z} \subseteq \mathbb{R}^k,$$

whose elements $z = (z_1, \ldots, z_k)$ are *production plans*. We follow the sign convention:

$$z_i \leq 0 \text{ (inputs)}, \qquad z_j \geq 0 \text{ (outputs)}.$$

Example. With two inputs and one output, $z = (-2, -3, 24) \in \mathcal{Z}$ means the firm uses 2 units of input 1 and 3 of input 2 to produce 24 units of output.

Remark. This sign convention makes the language flexible: "adding output" increases components of z that are outputs, while "using more inputs" decreases the corresponding (negative) components.

Prices and profits

Let $p \in \mathbb{R}^k$ be the *price vector*. The firm's (static) profit from plan z is

$$\pi(p, z) = p \cdot z = \sum_{i=1}^{k} p_i z_i.$$

With outputs priced $p_j \geq 0$ and inputs priced $p_i \geq 0$ (remember $z_i \leq 0$ for inputs), outputs raise profit and inputs reduce it.

The firm's problem and the profit function

Given p, the set of profit-maximizing plans is

$$Z(p) \ := \ \arg\max_{z \in \mathcal{Z}} \ p \cdot z,$$

which can be a *correspondence* (multiple maximizers) or empty. The associated profit function is

$$\pi(p) \ := \ \sup\{\, p \cdot z \ : \ z \in \mathcal{Z} \,\} \quad \text{(well-defined once } \mathcal{Z} \neq \varnothing\text{)},$$

and may take value $+\infty$ if technology allows arbitrarily profitable scale at prices p.

Remark (Existence (minimal conditions))**.** If $\mathcal{Z}$ is nonempty, closed, and profits are *bounded above* at p (no arbitrarily scalable free output in a priced direction), then $Z(p) \neq \varnothing$. Uniqueness generally needs strict concavity of the feasible frontier (or of π).

A two-dimensional picture

Consider one input $L \leq 0$ and one output $y \geq 0$. A convenient representation is

$$\mathcal{Z} \;=\; \{(L, y) \in \mathbb{R}^2 :\; y \leq \sqrt{-L},\; L \leq 0,\; y \geq 0\},$$

which says higher output requires (weakly) more input. The feasible set is the area under the curve $y = \sqrt{-L}$ for $L \leq 0$.

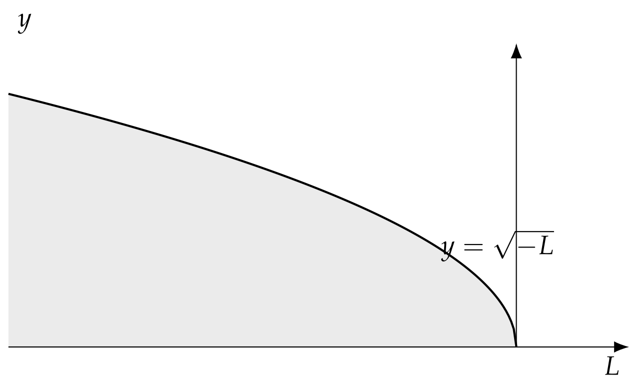


Figure 25: Technology with one input ($L \leq 0$) and one output ($y \geq 0$). The shaded area is the feasible set $\mathcal{Z}$.

Iso-profit lines. At prices (p_L, p_y) with $p_L, p_y > 0$, iso-profit lines in (L, y)-space have slope $-\frac{p_L}{p_y}$ (since $\pi = p_y\, y + p_L\, L$ with $L \leq 0$). Profit maximization picks a boundary point of $\mathcal{Z}$ where an iso-profit line is just tangent (or a corner like the origin if prices are unfavorable).

Takeaways:

- Technology as a set $\mathcal{Z} \subset \mathbb{R}^k$ (outputs ≥ 0, inputs ≤ 0) gives a clean language for firm decisions.
- The choice set at prices p is $Z(p) = \arg\max_{z \in \mathcal{Z}} p \cdot z$; $Z(p)$ can be multi-valued. The profit function is $\pi(p) = \sup_{z \in \mathcal{Z}} p \cdot z$.
- Existence needs nonemptiness/closedness and bounded profits at p; uniqueness needs curvature.

A warm-up example (CRS ray and (non)existence)

Let $k = 2$ with $z = (z_1, z_2)$, where $z_1 \leq 0$ is an input and $z_2 \geq 0$ an output. Consider the constant-returns technology

$$\mathcal{Z} \ = \ \{(z_1, z_2) \in \mathbb{R}^2 : \ z_1 \leq 0, \ z_2 = |z_1|\} = \{(-t, t) : \ t \geq 0\}.$$

At prices $p = (p_1, p_2)$ the firm solves $\max_{t \geq 0} p \cdot (-t, t) = t(p_2 - p_1)$. Hence

$$Z(p) = \begin{cases} \{(0,0)\} & \text{if } p_1 > p_2, \\ \{(-t,t) : t \geq 0\} & \text{if } p_1 = p_2, \\ \varnothing & \text{if } p_1 < p_2, \end{cases} \qquad \pi(p) = \begin{cases} 0 & \text{if } p_1 \geq p_2, \\ +\infty & \text{if } p_1 < p_2. \end{cases}$$

Lesson: if an output's price exceeds the "cost" of the input along a CRS ray, profits are unbounded and no maximizer exists.

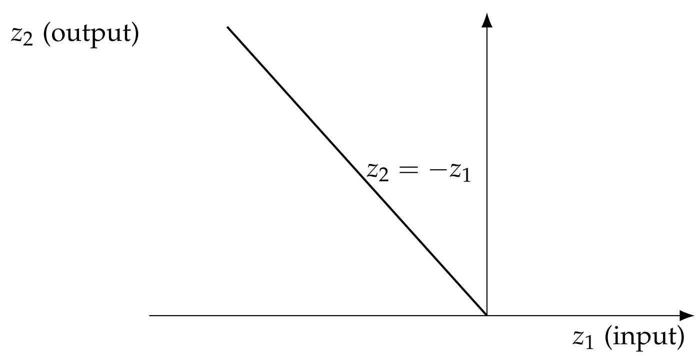


Figure 26: CRS ray. If $p_1 > p_2$, the maximizer is $(0, 0)$; if $p_1 < p_2$, profits are unbounded.

Existence, continuity, and convexity of the choice set

Proposition (Existence under compactness)**.** *If $\mathcal{Z}$ is nonempty and compact, then for every p the maximization* $\max_{z \in \mathcal{Z}} p \cdot z$ *admits a solution; i.e. $Z(p) \neq \varnothing$.*

Remark (Free disposal as a fallback)**.** If $0 \in \mathcal{Z}$ and the technology has *free disposal* (i.e. $z' \leq z$ componentwise and $z \in \mathcal{Z}$ imply $z' \in \mathcal{Z}$), then for any $p \geq 0$ the problem has at least the feasible plan $z = 0$ with $\pi(p, 0) = 0$.

Theorem (Berge's Maximum Theorem)**.** *If $\mathcal{Z}$ is nonempty and compact, then the argmax correspondence $Z(\cdot)$ is upper hemicontinuous (hence has a closed graph), and the profit function $\pi(p) = \max_{z \in \mathcal{Z}} p \cdot z$ is continuous in p.*

Proposition (Convex choice at fixed prices)**.** *If $\mathcal{Z}$ is convex, then $Z(p)$ is convex for every p.* Proof. *If $z, z' \in Z(p)$ and $\lambda \in [0, 1]$, then $\tilde{z} = \lambda z + (1 - \lambda) z' \in \mathcal{Z}$ and $p \cdot \tilde{z} = \lambda\, p \cdot z + (1 - \lambda)\, p \cdot z' = \pi(p)$, so $\tilde{z} \in Z(p)$.* $\square$

Homogeneity and the law of supply (WAPM)

Proposition (Positive homogeneity). *For all $\lambda > 0$ and all p,*

$$Z(\lambda p) = Z(p) \qquad \text{and} \qquad \pi(\lambda p) = \lambda\, \pi(p).$$

Proposition (Weak axiom of profit maximization / law of supply). *If $z \in Z(p)$ and $z' \in Z(p')$, then*

$$(z - z') \cdot (p - p') \ \geq\ 0.$$

Proof. *From $z \in Z(p)$ we have $p \cdot z \geq p \cdot z'$; from $z' \in Z(p')$ we have $p' \cdot z' \geq p' \cdot z$. Adding yields $(p - p') \cdot (z - z') \geq 0$.* □

Corollary (One–price changes, inputs vs. outputs). *If only component j of the price vector changes from p_j to p'_j with $p'_j > p_j$, then*

$$(z'_j - z_j)(p'_j - p_j) \ \geq\ 0.$$

Hence for an output *j (quantities $z_j \geq 0$) we get $z'_j \geq z_j$ (more output when its price rises); for an* input *i (quantities $z_i \leq 0$) we get $z'_i \geq z_i$, i.e. the input is used* less *(moves toward zero).*

"Afriat for firms": rationalizing a finite dataset

Suppose we observe n price–choice pairs $\{(p^t, z^t)\}_{t=1}^n$.

Necessary condition (pairwise optimality). A *necessary* condition for profit maximization is

$$p^t \cdot z^t \ \geq\ p^t \cdot z^s \qquad \text{for all } t, s \in \{1, \ldots, n\}. \qquad (\star)$$

Sufficient condition and a rationalizing technology. If $(\star)$ holds, then the closed, convex set

$$\mathcal{Z}^* \ :=\ \left\{ z \in \mathbb{R}^k :\ p^t \cdot z \leq p^t \cdot z^t \text{ for all } t \right\}$$

(with free disposal added if desired: $\mathcal{Z}^* + \mathbb{R}^k_-$) *rationalizes* the data: each z^t solves $\max_{z \in \mathcal{Z}^*} p^t \cdot z$. If, moreover, all observed profits are nonnegative ($p^t \cdot z^t \geq 0$ for all t), then $0 \in \mathcal{Z}^*$ (consistent with free disposal and the idea that "doing nothing" is feasible).

What we can and cannot infer. From finite data we cannot identify the true technology nor its curvature; we can only assert that there *exists* a closed, convex (free-disposal) technology that makes the observations profit maximizing.

Takeaways.

- Compact $\mathcal{Z} \Rightarrow$ existence; Berge $\Rightarrow Z(\cdot)$ upper hemicontinuous and $\pi(\cdot)$ continuous.
- If $\mathcal{Z}$ is convex, $Z(p)$ is convex (mixtures of optimizers remain optimal).
- π and Z are positively homogeneous of degree 1 and 0, respectively.
- WAPM: $(z - z') \cdot (p - p') \geq 0$; in particular, raising an output price raises optimal output, and raising an input price lowers input usage.
- Afriat-for-firms: pairwise profit inequalities are necessary; they are also sufficient once we construct a rationalizing (closed, convex, free-disposal) $\mathcal{Z}^*$.