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# MICROECONOMICS 603 UNIVERSITY OF MICHIGAN

*Contents*

# *Class 1: What we have left from 601*

## *Review and Transition*

This first class revisits the key elements of consumer and firm theory that we covered in Econ 601, emphasizing the aspects that we will build on for general equilibrium analysis.

- **Consumer Theory:** Preferences, utility representation, and demand functions. We recall how complete, transitive, and continuous preferences can be represented by a continuous utility function $u : X \to \mathbb{R}$, and how monotonicity and convexity yield the standard shape of indifference curves.
- **The Firm:** Technology sets, production functions, cost and profit functions, and the duality between them. Firms choose input–output bundles to maximize profits given prices.
- **Bridging to 603:** Both consumers and firms are *price-takers* in competitive markets. Today we start from the individual optimization problems and move toward their interaction in equilibrium, which will be the main focus of 603.

**Remark** (Conceptual link)**.** In partial equilibrium, we studied individual optimization taking prices as given. In general equilibrium, prices themselves must adjust so that all markets clear simultaneously. Hence, equilibrium analysis "closes the model" by endogenizing prices.

## *Consumer problem*

We start from the familiar optimization problem:

$$\max_{x \in \mathbb{R}^n_+} u(x) \quad \text{s.t.} \quad p \cdot x \leq y.$$

Assume the solution is unique. Then the *Marshallian demand* is a well-defined function

$$x(p, y) : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \mathbb{R}^n_+.$$

**Remark.** Uniqueness ensures that $x(p, y)$ is a function rather than a correspondence. This assumption will be convenient later when we aggregate demand.

### *Firm problem*

Similarly, the firm solves

$$\max_{x \in X} \; p \cdot x,$$

where $X \subseteq \mathbb{R}^n$ represents the production set. The optimal supply is denoted by

$$s(p) \in X.$$

**Remark** (Partial order)**.** The production set $X$ is endowed with the *natural partial order* of $\mathbb{R}^n$: for $x, x' \in \mathbb{R}^n$, we say $x \geq x'$ if $x_i \geq x'_i$ for all $i$. This captures the idea of "more output and less input." Under this order, efficiency means that if $x$ is feasible, no $x' \geq x$ with $x' \neq x$ is feasible.

### *Monotonicity properties*

Consider a function $f : \mathbb{R}^n \to \mathbb{R}^n$. We say that $f$ is *monotone* if

$$(x - x') \cdot (f(x) - f(x')) \geq 0 \quad \forall x, x' \in \mathbb{R}^n.$$

If the inequality is reversed ($\leq 0$), the function is *monotonically decreasing*. This definition uses the inner product and the partial order above.[1]

[1] *Intuition for future me:* For the firm, monotonicity means that changing prices in a given direction produces a supply response in the same direction. The inverse concept applies to the demand function, which is decreasing.

### *Jacobian and local monotonicity*

Let $J(x)$ denote the Jacobian (matrix of first derivatives) of $f(x)$. Then

$$\frac{J(x) + J(x)^{\mathsf{T}}}{2}$$

is the *symmetric part* of the Jacobian. If this matrix is

- negative semidefinite (NSD), then $f$ is monotonically decreasing;
- positive semidefinite (PSD), then $f$ is monotonically increasing.

### *A stronger statement (firm side)*

Let $s : \mathbb{R}^n_{++} \to \mathbb{R}^n_{+}$ be the firm's supply function. Then $s$ is the result of profit maximization for some production set $X$ if and only if:

(1) $s(p)$ is homogeneous of degree zero:

$$s(\lambda p) = s(p) \quad \forall \lambda > 0;$$

(2) The Jacobian $J(p)$ is symmetric and positive semidefinite:

$$(p - p') \cdot (s(p) - s(p')) \geq 0.$$

**Remark.** Condition (1) follows from the fact that only relative prices matter for profit maximization. Condition (2) encodes the law of supply: as prices rise, the value supplied does not decrease.

## *Integrability and Expenditure Minimization*

### *Integrability Theorem*

A function $d : \mathbb{R}^{n+1}_{++} \to \mathbb{R}^n_+$, written as $d(p, y)$, is the result of utility maximization if and only if:

(1) **Budget constraint:** $p \cdot d(p, y) = y$ for all $(p, y)$.

(2) **Homogeneity:** $d(\lambda p, \lambda y) = d(p, y)$ for all $\lambda > 0$.

(3) **Symmetry and definiteness:** The matrix constructed as

$$\left( \frac{\partial d_i(p, y)}{\partial p_j} + d_j(p, y) \frac{\partial d_i(p, y)}{\partial y} \right)_{i,j=1}^{n}$$

is symmetric and negative semidefinite (NSD).

**Remark.** These conditions guarantee that a demand function can be rationalized by some utility function. The third condition (symmetry and NSD) is the analogue of compensated monotonicity: it ensures that *compensated price changes* (holding utility constant) yield predictable, symmetric responses.

### *Expenditure Minimization Problem*

$$\min_{x \in \mathbb{R}^n_+} \; p \cdot x \quad \text{s.t.} \quad u(x) \geq \bar{u}.$$

This problem finds the *cheapest bundle* that attains a given utility level $\bar{u}$. The solution defines the *Hicksian demand*:

$$x^h(p, \bar{u}),$$

and the minimized value is the *expenditure function*:

$$e(p, \bar{u}) = p \cdot x^h(p, \bar{u}).$$

**Remark.** Geometrically, $x^h(p, \bar{u})$ is the point of tangency between the budget line and the indifference curve $u(x) = \bar{u}$. It represents a dual problem to the consumer's utility maximization.

### *Properties of the expenditure function*

**Proposition.** *$e(p, \bar{u})$ is concave in $p$.*

*Proof Sketch.* For any $\lambda \in [0,1]$ and price vectors $p, p'$,[2]

$$\lambda e(p, \bar{u}) + (1-\lambda) e(p', \bar{u}) \leq e(\lambda p + (1-\lambda) p', \bar{u}).$$

□

[2] *Intuition for future me:* Since expenditure is linear in prices for each fixed bundle, minimizing over bundles yields a concave envelope in $p$.

**Proposition** (Shephard's Lemma)**.**

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = x_i^h(p, \bar{u}).$$

**Remark.** This follows directly from the Envelope Theorem. Intuitively, the derivative of the minimal expenditure with respect to the price of good $i$ tells us how much of that good is being purchased at the optimum.

### *Further properties*

**Proposition.** *The Hessian $\nabla_p^2 e(p, \bar{u})$ is symmetric and negative semidefinite.*

**Remark.** This implies that the Jacobian of the Hicksian demand is symmetric and NSD:

$$\frac{\partial x_i^h(p, \bar{u})}{\partial p_j} = \frac{\partial^2 e(p, \bar{u})}{\partial p_i \partial p_j} = \frac{\partial x_j^h(p, \bar{u})}{\partial p_i} \quad \text{(by Young's Theorem).}$$

### *The Slutsky Equation*

$$\frac{\partial d_i(p, y)}{\partial p_j} = \frac{\partial x_i^h(p, \bar{u})}{\partial p_j} - d_j(p, y) \frac{\partial d_i(p, y)}{\partial y}.$$

- The first term is the *substitution effect*: response to price changes keeping utility fixed.

- The second term is the *income effect*: adjustment due to the change in real income.

*Proof Idea.* Since at the optimum $x_i^h(p, u(p, y)) = d_i(p, y)$, differentiating with respect to $p_j$ and applying Shephard's Lemma yields

$$\frac{\partial x_i^h(p, \bar{u})}{\partial p_j} = \frac{\partial d_i(p, y)}{\partial p_j} + \frac{\partial d_i(p, y)}{\partial y} \frac{\partial e(p, \bar{u})}{\partial p_j}.$$

Substituting $\frac{\partial e}{\partial p_j} = x_j^h = d_j$ gives the result. □

**Remark.** The Slutsky equation decomposes the total effect of a price change on demand into the part due to substitution along an indifference curve and the part due to the change in purchasing power. The matrix of substitution effects (the Slutsky matrix) is symmetric and negative semidefinite.

# *Class 2: Pareto Efficiency*

The notion of efficiency that we are going to use is *Pareto efficiency*. We are going to distinguish it from "Pareto optimality" that will need the existence of a welfare function.
**Primitives.**

- **Goods:** $\ell = 1, 2$.
- **Initial endowment:** $\omega = (\omega_1, \dots, \omega_L) \in \mathbb{R}^L_+$.
- **Production set:** $Y \subseteq \mathbb{R}^L$. A vector $y \in Y$ is a *production plan*; typically $y_\ell < 0$ (input) and $y_\ell > 0$ (output).

**Definition** (Post–production feasibility set)**.** Let $Z \subseteq \mathbb{R}^L$ be defined by the Minkowski sum

$$Z = \{\omega\} + Y \;=\; \{\, x \in \mathbb{R}^L \mid x = \omega + y \text{ for some } y \in Y \,\}.$$

$Z$**:** the set feasible for the economy after production.
*Assumption? Properties?*

**Consumers (that also supply labour).**

- $i = 1, 2, \dots, I$.
- **Consumption set:** $X_i \subseteq \mathbb{R}^L_+$. Consumer $i$'s *consumption plan* is $x_i \in X_i$.
- **Economy-wide plan:** $X = (x_1, x_2, \dots, x_I) \in \mathbb{R}^{I\times L}_+$.
- **Preferences:** $\succeq_i$ on $X_i$, complete and transitive.

**Definition** (Feasibility)**.** A consumption plan $x = (x_i)_{i\in I}$ is *feasible* if

$$\sum_{i\in I} x_i \in Z \qquad \Longleftrightarrow \qquad Z = \{\omega\} + Y.$$

**Definition** (Pareto efficiency)**.** A feasible consumption plan $x$ is *Pareto-efficient* if there is no other feasible plan $\hat{x}$ such that

$$\hat{x}_i \succeq_i x_i \;\; \forall i, \qquad \text{and} \qquad \hat{x}_j \succ_j x_j \;\text{ for some } j.$$

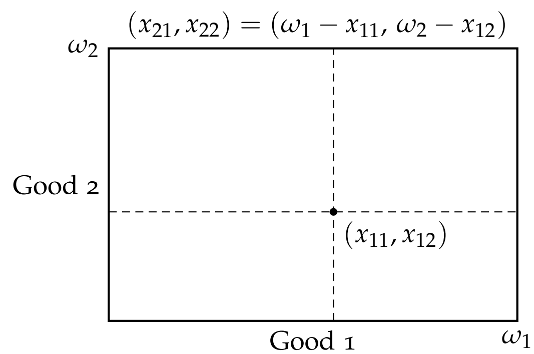
There should be a better way to define it.

No notion of interpersonal comparisons → very interesting to discuss.

How satisfied consumers are? → central Q. of the course.

## *Pure Exchange and the Edgeworth Box*

Often we will focus on an exchange economy where $Y = \{0\}$ (no production). Suppose $L = I = 2$ (a $2 \times 2$ economy). Then every feasible allocation lies in the *Edgeworth box* of width $\omega_1$ and height $\omega_2$.




Edgeworth box ($2 \times 2$ exchange). Every feasible consumption plan lies inside the box.

Every feasible consumption plan lies in the box.

## *Planner's Program and Tangency*

Let the aggregate endowment be $\omega = (\omega_1, \omega_2)$, and utilities $u_1(x_{11}, x_{12})$ and $u_2(x_{21}, x_{22})$. A standard Pareto program is

$$\max_{x_{11}, x_{12}} \; u_1(x_{11}, x_{12}) \quad \text{s.t.} \quad u_2(\omega_1 - x_{11},\, \omega_2 - x_{12}) \;\geq\; \bar{u}.$$

Its Lagrangian is

$$\mathcal{L} = u_1(x_{11}, x_{12}) - \lambda\Big(\,\bar{u} - u_2(\omega_1 - x_{11},\, \omega_2 - x_{12})\Big).$$

The first-order conditions (for interior solutions) are

$$\frac{\partial \mathcal{L}}{\partial x_{11}} = \frac{\partial u_1}{\partial x_{11}} - \lambda\,\frac{\partial u_2}{\partial x_{21}} = 0, \qquad \frac{\partial \mathcal{L}}{\partial x_{12}} = \frac{\partial u_1}{\partial x_{12}} - \lambda\,\frac{\partial u_2}{\partial x_{22}} = 0,$$

which imply (again, for interior solutions and differentiability)

$$\frac{\partial u_1/\partial x_{11}}{\partial u_1/\partial x_{12}} \;=\; \frac{\partial u_2/\partial x_{21}}{\partial u_2/\partial x_{22}} \qquad \Longleftrightarrow \qquad MRS_1 = MRS_2.$$

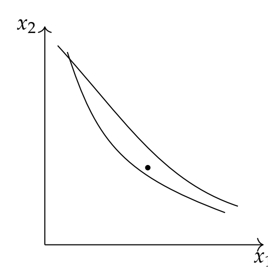
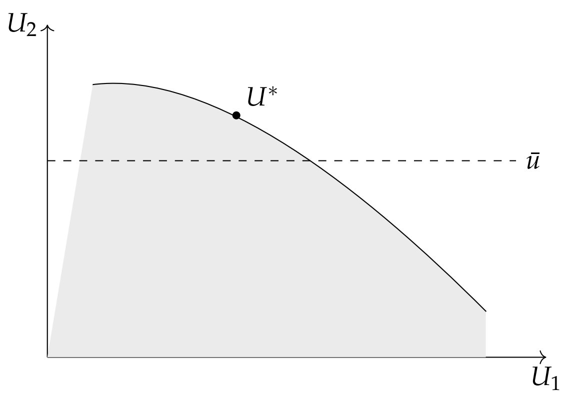



Figure 2: Interior PE: indifference curves tangent $\Rightarrow MRS_1 = MRS_2$.

**Remark.** *Can we have multiple solutions?* Yes. Without strict convexity of preferences (e.g., kinks or linear segments), the efficient set can include a whole segment of allocations, so the planner's problem at a given $\bar{u}$ need not have a unique maximizer.

## *Utility Possibility Set*




Utility possibility set (shaded) and frontier; maximizing $U_1$ subject to $U_2 \geq \bar{u}$.

# *Class 3: Walrasian Equilibrium*

The setup will be slightly more elaborated than in Pareto efficiency.
**Primitives.**

- Goods: $\ell = 1, \dots, L$.
- Consumers: $i = 1, \dots, I$ with consumption set $X_i \subseteq \mathbb{R}^L_+$ and preferences $\succeq_i$ (complete and transitive).
- Endowments: $\omega_i \in \mathbb{R}^L_+$, aggregate $\omega = \sum_{i=1}^I \omega_i$.
- Firms: $j = 1, \dots, J$ with production sets $Y_j \subseteq \mathbb{R}^L$.
- Ownership shares: $\theta_i^j \in [0,1]$ with $\sum_{i=1}^I \theta_i^j = 1$ for each $j$.
- Aggregate production set: $Y = \sum_{j=1}^J Y_j$ (Minkowski sum).

"Private ownership economy". There can be more.

**Prices and allocations.** A price vector is $p \in \mathbb{R}^L$. An allocation is $(x_1, \dots, x_I; y_1, \dots, y_J)$ with $x_i \in X_i$ and $y_j \in Y_j$. Consumer $i$'s budget set at $(p, y)$:

$$B_i(p,y) \equiv \left\{ x_i \in X_i \;\middle|\; p \cdot x_i \le p \cdot \omega_i + \sum_{j=1}^J \theta_i^j \, p \cdot y_j \right\}.$$

**Definition** (Walrasian equilibrium)**.** A price vector $p^*$ and an allocation $(x^*, y^*)$ are a WE if

(i) **Profit maximization:** $p^* \cdot y_j^* \ge p^* \cdot y$ for all $y \in Y_j$ and all $j = 1, \dots, J$.

(ii) **Utility maximization:** $x_i^* \in B_i(p^*, y^*)$ and $x_i^* \succeq_i x_i$ for all $x_i \in B_i(p^*, y^*)$, for each $i$.

(iii) **Market clearing:** $\sum_{i=1}^I x_i^* = \sum_{i=1}^I \omega_i + \sum_{j=1}^J y_j^*$.

Essence: find supply (i), find demand (ii) and then equalize them (iii).

**Remark** (Price normalization)**.** If $(p^*, x^*, y^*)$ is a WE and $\lambda > 0$, then $(\lambda p^*, x^*, y^*)$ is also a WE. If $p_\ell^* > 0$, we may normalize $p_\ell^* = 1$ w.l.o.g.

**Proposition** (Budget exhaustion & Walras' law)**.** *If all $\succeq_i$ are locally non-satiated and (ii) holds at $p^*$, then*

$$p^* \cdot \sum_{i=1}^I x_i^* \;=\; p^* \cdot \sum_{i=1}^I \omega_i \;+\; p^* \cdot \sum_{j=1}^J y_j^*.$$

**Remark.** Proof sketch: with LNS, each consumer spends the whole budget: $p^* \cdot x_i^* = p^* \cdot \omega_i + \sum_j \theta_i^j \, p^* \cdot y_j^*$. Sum over $i$ and use $\sum_i \theta_i^j = 1$.

**Proposition** (Redundancy of one market)**.** *If $\succeq_i$ are locally non-satiated, (ii) holds, and*

$$\sum_{i=1}^{I} x_{i\ell}^* = \sum_{i=1}^{I} \omega_{i\ell} + \sum_{j=1}^{J} y_{j\ell}^* \quad \textit{for } \ell = 1, \ldots, L-1,$$

*then for any good L with $p_L^* \neq 0$,*

$$\sum_{i=1}^{I} x_{iL}^* = \sum_{i=1}^{I} \omega_{iL} + \sum_{j=1}^{J} y_{jL}^*.$$

*Appendix: Walrasian Equilibrium Cookiecutter*

**Seven rules that solve 90% of these problems**

1. **Pick the program & normalize.** Pareto (planner) vs. WE (markets). Set a numeraire (e.g., $p_1 = 1$).

2. **Collapse dimension by feasibility.** In $2 \times 2$, write $x_{11} = \omega_1 - x_{21}$ and $x_{12} = \omega_2 - x_{22}$; reduce to 1 variable whenever one constraint is linear (e.g., $2x_{21} + x_{22} = \bar{u}_2$).

3. **Interior Pareto = tangency of MRS.**
$$\frac{MU_{1,1}}{MU_{1,2}} = \frac{MU_{2,1}}{MU_{2,2}} \quad \Rightarrow \quad \text{ratio-of-ratios (contract curve).}$$
If utility has linear/quasi-linear parts, *solve 1D*, check $u_1' = 0$; if it falls outside the range, the solution is *edge*.

4. **Identical Cobb–Douglas $\Rightarrow$ proportional split.** With $u_i = x_{i1}x_{i2}$: $x_{i\ell} = \alpha_i \cdot (\text{total of good } \ell)$. Then I chose separate production (see 5).

5. **Separate production: I chose $y$ to maximize the aggregator that everyone values.** With $Y: \ y_2 = \sqrt{-y_1}$ and $\omega = (\omega_1, 0)$:
$$\max_{0 \le y_1 \le \omega_1} (\omega_1 - y_1)\sqrt{y_1} \ \Rightarrow \ y_1^\star = \omega_1/3, \quad y_2^\star = \sqrt{\omega_1/3}.$$

6. **Minimum WE pipeline.** (i) Signature: $\pi(p) = \max_{y \in Y} p \cdot y$ (with $y_2 = \sqrt{-y_1}$: $y_1^\star = (p/2)^2$, $y_2^\star = p/2$, $\pi = p^2/4$). (ii) Income: $m_i = p \cdot \omega_i + \sum_j \theta_i^j \pi_j$. (iii) Demand CD with equal exponents: each $i$ spends *half* of $m_i$ on each good. (iv) **Clear only one market** (Walras): the other closes on its own.

7. **Two laws that save you steps.** LNS $\Rightarrow$ *exhausted* budgets. Walras' Law $\Rightarrow$ *one* equilibrium equation is redundant.

**Remark** (Quick pitfalls)**.** (1) Include firm profits exactly once in income: $m_i = p \cdot \omega_i + \sum_j \theta_i^j \pi_j(p)$; never add profits to the goods resource constraint. (2) Use only $L - 1$ market-clearing equations—Walras' Law makes the last one redundant after price normalization. (3) FOCs give candidates only: enforce $x_{i\ell} \ge 0$ and any auxiliary constraints (e.g., $u_2 \ge \bar{u}_2$); if violated, the solution lies on the boundary.

## *Exercises*

1. Consider an exchange economy that consists of two consumers $i = 1, 2$. There are two consumption goods $\ell = 1, 2$. The economy has a total endowment of 5 units of good 1 and 5 units of good 2. Both consumers have consumption set $\mathbb{R}^2_+$. Consumer 1 has preferences that can be represented by the utility function

$$u_1(x_{1,1}, x_{1,2}) \;=\; x_{1,1}\,(x_{1,2})^2.$$

   Consumer 2 has preferences that can be represented by the utility function

$$u_2(x_{2,1}, x_{2,2}) \;=\; (x_{2,1})^2\,x_{2,2}.$$

   Which consumption plans are Pareto efficient?

2. Repeat Question 1 but now assume that

$$u_1(x_{1,1}, x_{1,2}) \;=\; x_{1,1} + \sqrt{x_{1,2}} \qquad \text{and} \qquad u_2(x_{2,1}, x_{2,2}) \;=\; 2x_{2,1} + x_{2,2}.$$

3. Suppose an economy has initial endowment $\omega = (5, 0)$ and suppose the economy's production set is

$$Y \;=\; \{\,(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \leq 0 \text{ and } y_2 = \sqrt{-y_1}\,\}.$$

   There are two consumers. Both consumers have consumption set $\mathbb{R}^2_+$. Consumer 1 has preferences that can be represented by the utility function

$$u_1(x_{1,1}, x_{1,2}) \;=\; x_{1,1}\,x_{1,2}$$

   and consumer 2's preferences are represented by

$$u_2(x_{2,1}, x_{2,2}) \;=\; x_{2,1}\,x_{2,2}.$$

   Which consumption plans are Pareto efficient in this economy?

4. For the economy in Question 3 find a Walrasian Equilibrium. Assume that both consumers have initial endowment $\omega_i = (2.5, 0)$, and that there is one firm. This firm's production set is given by the set $Y$ in Question 3. Assume that both consumers own shares 0.5 of the firm.

5. Consider an exchange economy that consists of 2 consumers $i = 1, 2$. There are two consumption goods $\ell = 1, 2$. The economy has a total endowment of 5 units of good 1 and 8 units of good 2. Both consumers have consumption set $\mathbb{R}^2_+$. Consumer 1 regards the two goods as perfect substitutes. Her preferences are represented by the utility function

$$u_1(x_{1,1}, x_{1,2}) = x_{1,1} + x_{1,2}.$$

Consumer 2 regards the two goods as complements. His preferences are represented by the utility function

$$u_2(x_{2,1}, x_{2,2}) = \min\{x_{2,1}, x_{2,2}\}.$$

Which consumption plans are Pareto efficient?

6. Consider an exchange economy that consists of $n \geq 2$ consumers. There are two consumption goods $\ell = 1, 2$. The economy has a total endowment of 10 units of good 1 and 20 units of good 2. All consumers have consumption set $\mathbb{R}^2_+$. The consumers have identical preferences represented by

$$u_i(x_{i,1}, x_{i,2}) = x_{i,1}\,(x_{i,2})^2 \quad \text{for } i = 1, 2, \ldots, n.$$

Which consumption plans are Pareto efficient? Which consumption plans maximize the sum of the consumers' utility? (For the second question, it may be helpful to start with the case $n = 2$.)

7. Consider an economy with two consumers. Both consumers have consumption set $\mathbb{R}^2_+$. Consumer 1 has preferences represented by

$$u_1(x_{1,1}, x_{1,2}) = x_{1,1}\,x_{1,2},$$

and consumer 2's preferences are represented by

$$u_2(x_{2,1}, x_{2,2}) = x_{2,1}\,x_{2,2}.$$

Consumer 1 has initial endowment $\omega_1 = (5, 0)$ whereas consumer 2 has initial endowment $\omega_2 = (0, 0)$. Consumer 2 owns all of the firm. There is one firm with production set

$$Y = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \leq 0 \text{ and } y_2 \leq 2\sqrt{-y_1}\}.$$

Find all Walrasian equilibria of this economy.

## *More complicated exercises*

**P1. Existence of Walrasian Equilibrium via Kakutani (pure exchange).** Let an exchange economy have $I \geq 2$ consumers, $L \geq 2$ goods, aggregate endowment $\Omega \gg 0$, and for each $i$, a continuous, strictly convex, locally non-satiated preference relation represented by a utility $u_i : \mathbb{R}^L_+ \to \mathbb{R}$. Define the aggregate excess demand $z(p) = \sum_{i=1}^I x_i(p, p \cdot \omega_i) - \Omega$ with prices $p \in \Delta = \{p \in \mathbb{R}^L_+ : \sum_\ell p_\ell = 1\}$.

(a) Prove: (i) $z$ is continuous on the simplex $\Delta$, (ii) $p \cdot z(p) = 0$ (Walras' Law), (iii) $z(\cdot)$ is homogeneous of degree 0, (iv) boundary behavior: if $p_\ell \downarrow 0$ then $z_\ell(p) \to +\infty$ under local non-satiation and $\Omega \gg 0$.

(b) Construct the excess demand correspondence with price truncation $Z(p) = \text{argmax}\{q \cdot z(p) : q \in \Delta\}$ and show the hypotheses of Kakutani's fixed-point theorem hold for an appropriate correspondence, yielding $p^* \in \Delta$ with $z(p^*) \leq 0$ componentwise.

(c) Conclude existence of a Walrasian equilibrium $(p^*, x^*)$, giving a precise selection argument from individual demands to allocations and verifying feasibility and optimality.

**P2. Negishi Program and Welfare Theorems (pure exchange).** Fix $\lambda \in \Delta_I^\circ = \{\lambda \in \mathbb{R}^I_{++} : \sum_i \lambda_i = 1\}$. Consider the social planner:

$$\max_{\{x_i\}_{i=1}^I} \sum_{i=1}^I \lambda_i u_i(x_i) \quad \text{s.t.} \quad \sum_{i=1}^I x_i = \Omega, \; x_i \in \mathbb{R}^L_+.$$

(a) Prove existence of a solution and derive the KKT conditions. Show that at any interior solution $\{x_i^\lambda\}$ there exists $p^\lambda \gg 0$ supporting $\{x_i^\lambda\}$ with $\text{MRS}_i(x_i^\lambda) = p^\lambda$ for all $i$.

(b) Define the Negishi map $T : \lambda \mapsto$ (vector of transfers making each $x_i^\lambda$ budget-feasible at $p^\lambda$). Prove continuity of $T$ and deduce existence of $\lambda^*$ with zero net transfers. Conclude: any competitive equilibrium allocation solves the planner's problem for some $\lambda^*$, and conversely.

**P3. Core–Walras Equivalence under Replication.** Consider the $\kappa$-replica economy with $\kappa \in \mathbb{N}$.

(a) Prove that the set of core allocations $\mathcal{C}^\kappa$ shrinks as $\kappa$ grows (i.e., $\mathcal{C}^{\kappa+1} \subseteq \mathcal{C}^\kappa$) under convex preferences and free disposal.

(b) Show that $\bigcap_{\kappa=1}^\infty \mathcal{C}^\kappa$ equals the set of Walrasian equilibrium allocations (Edgeworth's conjecture) by constructing equal-treatment allocations and using supporting price hyperplanes.

**P4. Gross Substitutes and Uniqueness.** Let aggregate excess demand $z : \Delta \to \mathbb{R}^L$ satisfy Walras' Law and GS: for any $p, p'$ with $p'_j > p_j$ and $p'_{-j} = p_{-j}$, we have $z_k(p') \geq z_k(p)$ for all $k \neq j$.

(a) Prove that any two Walrasian equilibria have comparable price vectors (lattice property) and that the set of equilibrium prices is a lattice interval.

(b) Deduce global uniqueness of the relative price vector if $z(\cdot)$ satisfies GS and the value of excess demand for at least one good is strictly decreasing in its own price.

(c) Establish monotone comparative statics: if endowments of good $j$ increase, then equilibrium $p_j/p_{-j}$ weakly decreases.

**P5**. **Multiplicity with Leontief: full characterization.** Two goods, two consumers with $u_1(x) = \min\{ax_1, x_2\}$ and $u_2(x) = \min\{x_1, bx_2\}$, $a, b > 0$, aggregate endowment $\Omega \gg 0$.

(a) Characterize the set of Pareto efficient allocations analytically (no diagrams): derive the piecewise-linear contract set by complementary slackness on the Leontief constraints.

(b) Prove that the set of equilibrium *relative* prices is a nondegenerate interval if and only if a specific linear balance condition between $\Omega$ and $(a, b)$ holds. State and prove the necessary and sufficient condition.

**P6**. **Competitive Equilibrium with Convex Production and Profit Duality.** One firm with technology $Y \subset \mathbb{R}^L$ closed, convex, containing 0, and free disposal. Profit function $\pi(p) = \sup_{y \in Y} p \cdot y$; supply correspondence $Y^*(p) = \arg\max_{y \in Y} p \cdot y$.

(a) Prove: $\pi$ is convex and positively homogeneous, $Y^*(p) = \partial\pi(p)$ (subdifferential), and $p \cdot y = \pi(p)$ for all $y \in Y^*(p)$.

(b) Show existence of a competitive equilibrium in the economy with $I$ consumers (as in **P1**) and this firm, invoking standard conditions and a price truncation argument. Be explicit about the "no unbounded profit at zero prices" condition and where it is used.

**P7**. **Shapley–Folkman and Approximate Equilibria with Nonconvexities.** Let individual feasible sets $F_i \subset \mathbb{R}^L$ be compact (not necessarily convex). Define $F = \sum_{i=1}^{I} F_i$.

(a) State and prove the Shapley–Folkman lemma.

(b) Use it to show that for large $I$ every point in $F$ is within $O(1/I)$ (in Hausdorff distance) of the Minkowski sum of convex hulls $\sum_i \mathrm{co}(F_i)$.

(c) Deduce an $\varepsilon$-equilibrium existence result: with smooth utilities and small nonconvexities, there exists a price-allocation pair that is $\varepsilon(I)$-approximately competitive with $\varepsilon(I) \to 0$ as $I \to \infty$.

**P8**. **Regular Economies and Local Comparative Statics.** Assume a pure exchange economy is *regular*: the Jacobian of market excess demand with respect to prices (restricted to the price simplex) at equilibrium has full rank.

(a) Prove that equilibrium prices are locally unique and vary smoothly with endowments (Implicit Function Theorem), after quotienting out the homogeneity.

(b) Derive a closed-form expression (up to normalization) for $D_\omega p^*(\omega)$ in terms of the Slutsky matrices and income effects, and sign the comparative statics when all agents are gross substitutes.

**P9. Assignment Model and Existence with Indivisibilities.** There are $n$ agents, $n$ indivisible objects, and quasilinear utilities $u_i(x_i) = v_i(j) - p_j$ if agent $i$ consumes object $j$, $u_i(0) = 0$ otherwise.

(a) Prove existence of a Walrasian equilibrium via the lattice-theoretic approach (Kelso–Crawford) under gross substitutes valuations.

(b) Show that the set of equilibrium prices forms a lattice and the buyer-optimal price vector exists.

**P10. Production–Exchange with Non-differentiable Frontiers.** Two goods $\{1, 2\}$; one firm with $Y = \{(y_1, y_2) : y_1 \leq 0,\ y_2 \leq A\sqrt{-y_1}\} \cup \{(0, 0)\}$; two consumers with strictly convex, $C^1$ utilities and positive endowments.

(a) Prove the firm's supply correspondence is upper hemicontinuous and nonempty, compact-valued for all $p \gg 0$ despite the kink at $y_1 = 0$; compute it explicitly.

(b) Establish existence of competitive equilibrium and characterize the comparative statics of $p_2/p_1$ with respect to the technology parameter $A$.

**P11. Price Support of Pareto Optima (Separation).** Let $\bar{x} = (\bar{x}_i)_{i=1}^{I}$ be a Pareto efficient allocation with $\bar{x}_i \gg 0$ and strictly convex, monotone preferences.

(a) Prove that there exists $p \gg 0$ such that each $\bar{x}_i$ solves $\max\{u_i(x_i) : p \cdot x_i \leq p \cdot \bar{x}_i\}$.

(b) Give a proof based *only* on convex separation (no welfare theorems) by constructing a separating hyperplane between the feasible set and an appropriate upper contour set at $\bar{x}$.

**P12. Sonnenschein–Mantel–Debreu (structured exercise).** Let $Z : \Delta \to \mathbb{R}^L$ be continuous, homogeneous of degree 0, and satisfy Walras' Law.

(a) Show that for any $\varepsilon > 0$ there exists a finite exchange economy whose aggregate excess demand $z$ is within $\varepsilon$ (uniformly on $\Delta$) of $Z$.

(b) Deduce that equilibrium price sets of exchange economies can approximate those of arbitrary $Z$ (subject to the two restrictions), and discuss the implications for uniqueness/stability theory.

**P13. Taxation to Decentralize Planner Allocations (exact implementation).** Consider a convex production economy with one firm $Y$ and consumers $\{u_i, \omega_i\}_{i=1}^{I}$. Let $\hat{x}$ solve the Negishi planner for weights $\lambda \in \Delta_I^\circ$ and let $\hat{p}$ be a supporting price vector for $\hat{x}$.

(a) Construct *linear* taxes/subsidies $\tau \in \mathbb{R}^L$ on goods (or, equivalently, on firm outputs) and lump-sum transfers $T$ such that $(\hat{p} + \tau, \hat{x})$ is a competitive equilibrium.

(b) Prove exact implementation and identify the (non-)uniqueness of $(\tau, T)$ under budget balance constraints.

# *Class 4: The First Welfare Theorem*

We have $I$ consumers and a list of consumption plans

$$x = (x_1, \ldots, x_I) \in X_1 \times X_2 \times \cdots \times X_I$$

and production plans

$$y = (y_1, \ldots, y_J) \in Y_1 \times Y_2 \times \cdots \times Y_J.$$

An *allocation* is a pair $(x, y)$. It is *feasible* if

$$\sum_{i=1}^{I} x_i \;=\; \sum_{i=1}^{I} \omega_i \;+\; \sum_{j=1}^{J} y_j \quad \text{(vector equality)}.$$

"Very general correspondences, simpler than functions."
Think more in dimensions $L$ (goods) and $I$ (consumers).

**Definition** (Pareto efficiency)**.** A feasible allocation $(x, y)$ is Pareto-efficient if there is no other feasible $(\hat{x}, \hat{y})$ such that $\hat{x}_i \succeq_i x_i$ for all $i$ and $\hat{x}_k \succ_k x_k$ for at least one $k$.

Recall: Walrasian equilibrium from Class 3.

## *First Welfare Theorem (FWT)*

**Theorem.** *If each $\succeq_i$ is locally non-satiated and $(p^*, x^*, y^*)$ is a Walrasian equilibrium, then $(x^*, y^*)$ is Pareto-efficient.*

*Proof.* Suppose, towards a contradiction, that there is a feasible $(\hat{x}, \hat{y})$ with $\hat{x}_i \succeq_i x_i^*$ for all $i$ and $\hat{x}_k \succ_k x_k^*$ for some $k$.

*Step 1 (consumer side).* With local non-satiation and optimality of $x_i^*$ in $B_i(p^*, y^*)$, any bundle strictly preferred to $x_i^*$ cannot be affordable; hence

$$\hat{x}_i \succeq_i x_i^* \implies p^* \cdot \hat{x}_i \;\geq\; p^* \cdot x_i^* \quad \text{for all } i,$$

with a strict inequality for at least one consumer $k$.

"Then it is *inside* your budget set" ⇒ would contradict optimality.

*Step 2 (firm side).* Profit maximization implies

$$p^* \cdot \hat{y}_j \;\leq\; p^* \cdot y_j^* \quad \text{for all } j.$$

*Step 3 (aggregate).* Summing Step 1 and using feasibility of $(\hat{x}, \hat{y})$,

$$p^* \cdot \sum_{i=1}^{I} \hat{x}_i \;>\; p^* \cdot \sum_{i=1}^{I} x_i^* \;=\; p^* \cdot \sum_{i=1}^{I} \omega_i \;+\; p^* \cdot \sum_{j=1}^{J} y_j^*.$$

But feasibility also gives $\sum_i \hat{x}_i = \sum_i \omega_i + \sum_j \hat{y}_j$, hence

$$p^* \cdot \sum_{j=1}^{J} \hat{y}_j \; > \; p^* \cdot \sum_{j=1}^{J} y_j^*,$$

which contradicts Step 2 (profit maximization). Therefore $(x^*, y^*)$ is Pareto-efficient. □

**Remark.** The key ingredients are: (i) local non-satiation ⇒ budget exhaustion and "strictly better ⇒ strictly more expensive"; (ii) profit maximization; (iii) feasibility. No welfare function is used.

### *What equalities hold at a WE (interior, differentiable case)?*

In a $2 \times 2$ interior WE with differentiable preferences and technologies:

$$MRS_i \; = \; \frac{p_1}{p_2} \quad \text{for all } i \qquad \text{and} \qquad MRT \; = \; \frac{p_1}{p_2}.$$

Thus $MRS_1 = \cdots = MRS_I = MRT = \frac{p_1}{p_2}$.

Assume interior solutions & differentiability.

**Remark** (Firm-side condition)**.** For a differentiable single-output technology $f(\cdot)$ with input $k$ and numeraire input $\ell$, profit maximization yields the familiar FOC

$$\frac{\partial f}{\partial k} \Big/ \frac{\partial f}{\partial \ell} \; = \; \frac{p_k}{p_\ell},$$

which is the production-side analogue of $MRS_i = p_1/p_2$ on the consumer side.

# *Class 5: The Second Welfare Theorem*

The First Welfare Theorem implies that if $(p^*, x^*, y^*)$ is a Walrasian equilibrium, then $(x^*, y^*)$ is one PE allocation among (typically) many on the feasible frontier.

FWT only says: any Walrasian equilibrium (WE) allocation is Pareto efficient (PE). It does *not* claim uniqueness.

FWT: WE $\subseteq$ PE. It is silent about existence and uniqueness. PE is generally a whole contract curve (many points). WE may be unique under extra assumptions (e.g., strict convexity + gross substitutes), but multiplicity is common (e.g., Leontief example).

## *Necessary equalities in the interior, differentiable case*

Assume: (i) local non-satiation, (ii) interior choices, (iii) differentiable preferences and technologies, and (iv) competitive factor and output markets.

**Proposition** (Consumer-side tangency)**.** *If $x_i^*$ is interior and differentiable, then*

$$MRS_i(x_i^*) \;=\; \frac{p_1^*}{p_2^*} \qquad \text{for all consumers } i.$$

**Remark.** Local non-satiation ⇒ budget exhaustion; interiority + differentiability ⇒ FOCs are necessary, yielding $\text{MRS}_i = p_1/p_2$. Without interiority (corners) or at kinks, tangency is not necessary.

**Proposition** (Firm-side efficiency and common factor prices)**.** *Let each firm $j$ use inputs $k, \ell$ with differentiable technology. Under competitive input prices $(w_k, w_\ell)$ and interior cost-minimizing choices,*

$$MRTS^j_{k,\ell} \;=\; \frac{\partial f_j/\partial k}{\partial f_j/\partial \ell} \;=\; \frac{w_k}{w_\ell} \qquad \text{for all firms } j.$$

*Hence $MRTS^j_{k,\ell} = MRTS^{j'}_{k,\ell}$ for all $j, j'$.*

**Remark.** If two firms had MRTS's different at a common input bundle, we could reallocate a small amount of $k, \ell$ across them and raise total output with the same factor endowment ⇒ production inefficiency. Competitive factor prices rule this out by equalizing MRTS across firms.

Aggregate the firm side into a production possibilities frontier (PPF). At the competitive production plan $(y_1^*, y_2^*)$ the slope of the PPF equals the output price ratio,

$$\text{MRT}(y^*) \;=\; \frac{p_1^*}{p_2^*}.$$

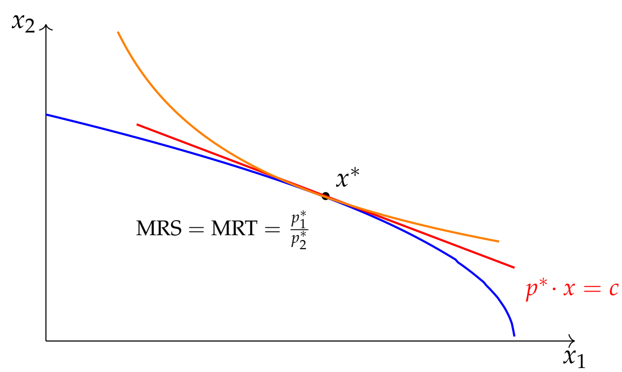
**Corollary** (Complete set of interior equalities)**.** *In a* $2 \times 2$ *interior WE with differentiability,*

$$\boxed{MRS_1 = \cdots = MRS_I \;=\; MRT \;=\; \frac{p_1^*}{p_2^*}}$$

*and, at the firm level,* $MRTS^j_{k,\ell} = \frac{w_k}{w_\ell}$ *for all* $j$*.*

Picture: the highest attainable indifference curve is *tangent* to the PPF. The common tangent has slope $-p_1/p_2$.

**Remark** (Why $(x^*, y^*)$ need not be unique?)**.** FWT selects *some* PE point; the PE set is generally a whole frontier (contract curve / utility possibility frontier). Different wealth distributions (endowments or lump-sum transfers) support different PE points while keeping the same technological PPF. Thus WE allocations are typically *not* unique.




Aggregate indifference (orange) is tangent to the PPF (blue) and to the supporting price line (red) at $x^*$.

**Remark** (Why can there be multiple WEs even with fixed fundamentals?)**.** Without additional structure, market-clearing price vectors may not be unique. Non–strict convexities (flat spots/kinks in indifference curves or technologies) or symmetries can generate multiple equilibria supported by different price ratios that all clear markets. Strict convexity helps uniqueness of *individual* demands, but does not guarantee uniqueness of the *aggregate* equilibrium.

**Remark** ("If moving to another equilibrium hurts someone, can we compensate? Think taxes.")**.** To move from one PE point to another without destroying efficiency, use *lump-sum* taxes/transfers (pure redistribution): they shift wealth but do not change marginal rates, so the equalities

$$\text{MRS} = \text{MRT} = \frac{p_1}{p_2}$$

remain intact. Distortionary taxes/subsidies on margins (e.g., ad valorem taxes on goods or inputs) generally break tangency by altering effective price ratios. **Preview (Second Welfare Theorem):** under convexity, any PE allocation can be decentralized as a WE after appropriate lump-sum redistribution.

**Remark** (When are the tangency conditions truly *necessary*?)**.** They require: (i) interior choices, (ii) differentiability (no kinks), and (iii) price-taking behavior. With corners or kinks, optimality may hold with inequalities (supporting hyperplanes), not equalities; then MRS and MRT equalities are not necessary, only the usual Kuhn–Tucker conditions are.

## *Taxes and transfers*

**Definition** (System of lump-sum transfers)**.** A *system of transfers* is a vector $T = (T_1, \ldots, T_I)$ with

$$\sum_{i=1}^{I} T_i \;=\; 0.$$

Each $T_i$ is a lump-sum tax ($T_i < 0$) or transfer ($T_i > 0$).

Lump-sum = redistributive but non-distortionary: it shifts wealth without changing marginal rates or effective prices.

**Definition** (WE with transfers)**.** Given ownership shares $\{\theta_{ij}\}_{i,j}$ (with $\sum_i \theta_{ij} = 1$ for each $j$), a triple $(p^*, x^*, y^*)$ is a *Walrasian equilibrium with transfers* $T$ if:

1. **Firms maximize profits:** for each $j$, $y_j^* \in Y_j$ solves $y \in Y_j \mapsto p^* \cdot y$.
2. **Consumers maximize utility:** for each $i$, $x_i^* \in X_i$ is $\succeq_i$-maximal subject to
$$p^* \cdot x_i \;\le\; p^* \cdot \omega_i \;+\; \sum_{j=1}^{J} \theta_{ij}\, p^* \cdot y_j^* \;+\; T_i.$$
3. **Feasibility (market clearing):** $\sum_{i=1}^{I} x_i^* \;=\; \sum_{i=1}^{I} \omega_i \;+\; \sum_{j=1}^{J} y_j^*$.

**Remark** (FWT is robust to lump-sum redistribution)**.** With local nonsatiation, the First Welfare Theorem applies verbatim to WE with transfers: $(x^*, y^*)$ is Pareto efficient. Intuition: transfers shift budgets but leave all marginal conditions MRS = MRT = price ratio intact.

Contrast: commodity/input taxes change *effective* prices and typically break tangency, i.e., MRS $\neq$ MRT.

## *Second Welfare Theorem (SWT)*

**Theorem** (SWT, decentralization by transfers)**.** *If $(x^*, y^*)$ is a Pareto-efficient allocation and the following assumptions hold, then there exist a price* *vector $p^* \gg 0$ and a transfer system $T$ (with $\sum_i T_i = 0$) such that $(p^*, x^*, y^*)$ is a Walrasian equilibrium with transfers $T$.*

**Remark.** The set of assumptions is:

1. **Convex consumption sets:** $X_i \subseteq \mathbb{R}^L_+$ is convex for all $i$.
2. **Convex preferences:** for all $i$, if $x'_i \succeq_i x_i$ and $\hat{x}_i \succeq_i x_i$, then $\lambda x'_i + (1-\lambda)\hat{x}_i \succeq_i x_i$ for all $\lambda \in [0,1]$.
3. **Convex production sets:** $Y_j$ is convex for all $j$.
4. **Regularity:** preferences are continuous and locally non-satiated.

Economic meaning: any PE point on the frontier can be implemented competitively after a pure wealth redistribution.

**Remark** (Sketch of the proof idea)**.** PE means there is no feasible improvement relative to $(x^*, y^*)$. Under convexity, a supporting hyperplane (separating hyperplane theorem) exists at the feasible set "seen through" agents' upper contour sets. Its normal is a price vector $p^*$. Lump-sum transfers then adjust individual budgets so that each $x^*_i$ is affordable at $p^*$. Competitive profit maximization and feasibility complete the decentralization.

**Remark** (What if convexity fails? (answering the orange note))**.** Without convexity (in preferences or technologies), the supporting-hyperplane argument can break:

- Some PE allocations *cannot* be decentralized by any common linear prices $\Rightarrow$ SWT may fail.
- Existence/uniqueness of WE can also be jeopardized (e.g., increasing returns, "holes" in indifference maps).

So we do not "rule out" all WEs, but we *do* lose the general PE $\Rightarrow$ WE (with transfers) implication.

**Remark** (Policy intuition)**.** Lump-sum $T$ shifts wealth across consumers to target the desired PE. Because $T$ does not alter marginal prices, we preserve MRS $=$ MRT $= \frac{p_1}{p_2}$ at the implemented allocation. Distortionary taxes would generally move the allocation *off* the frontier.

## *Separating hyperplane proof of the Second Welfare Theorem*

**Definition** (Aggregate feasibility set)**.** Let

$$R \;\equiv\; \left\{ r \in \mathbb{R}^L_+ \;\middle|\; r = \sum_{i=1}^{I} \omega_i \;+\; \sum_{j=1}^{J} y_j,\ y_j \in Y_j\ \forall j \right\}.$$

That is, $R$ collects all aggregate quantity vectors attainable from endowments and production. If each $Y_j$ is convex, then $R$ is convex.

**Definition** ("Better-than-$x^*$" aggregate set)**.** Given the target PE allocation $(x^*, y^*)$, define

$$V \;\equiv\; \Big\{ v \in \mathbb{R}^L_+ \;\Big|\; v = \sum_{i=1}^{I} x_i,\; x_i \in X_i,\; x_i \succeq_i x_i^* \;\forall i,\; x_k \succ_k x_k^* \text{ for some } k \Big\}.$$

If each $X_i$ is convex and $\succeq_i$ is convex, then $V$ is convex.

**Lemma** (Disjoint convex sets at PE)**.** *If $(x^*, y^*)$ is Pareto efficient, then $R \cap V = \varnothing$. Moreover, $R$ and $V$ are nonempty, closed, and convex under the standard assumptions (convex $X_i$, convex $Y_j$, continuous LNS preferences).*

PE means: *no* feasible allocation makes everyone weakly better and someone strictly better. Hence $R$ (feasible sums) and $V$ ("better-than-$x^*$") do not intersect.

**Theorem** (Supporting/separating hyperplane)**.** *Because $R$ and $V$ are nonempty, convex, and disjoint, there exist $p^* \in \mathbb{R}^L$, $p^* \neq 0$, and $c \in \mathbb{R}$ such that*

$$p^* \cdot r \;\le\; c \quad \forall r \in R, \qquad \text{and} \qquad p^* \cdot v \;\ge\; c \quad \forall v \in V.$$

*We may normalize $p^* \gg 0$ under LNS.*

**Remark** (Economic interpretation)**.** The hyperplane $p^* \cdot x = c$ is a *supporting price system*: it price-supports the PE frontier at $\sum_i x_i^*$. Points feasible to the economy lie (weakly) below, and points that (weakly) improve everyone lie (weakly) above; hence the frontier at $x^*$ is a tangency.

**Proposition** (Constructing lump-sum transfers to decentralize $(x^*, y^*)$)**.** *Let ownership shares $\{\theta_{ij}\}$ satisfy $\sum_i \theta_{ij} = 1$ for each $j$. Define*

$$\boxed{T_i \;\equiv\; p^* \cdot x_i^* \;-\; p^* \cdot \omega_i \;-\; \sum_{j=1}^{J} \theta_{ij}\, p^* \cdot y_j^*} \qquad (i = 1, \ldots, I).$$

*Then $\sum_i T_i = 0$, and $(p^*, x^*, y^*)$ is a Walrasian equilibrium with transfers $T$.*

*Verification.* **(i) Firms.** Since $p^*$ supports $R$ at $r^* = \sum_i \omega_i + \sum_j y_j^*$, each $y_j^*$ is profit-maximizing at prices $p^*$ (otherwise we could move inside $R$ and strictly raise $p^* \cdot r$, contradicting support).

**(ii) Consumers.** By construction,

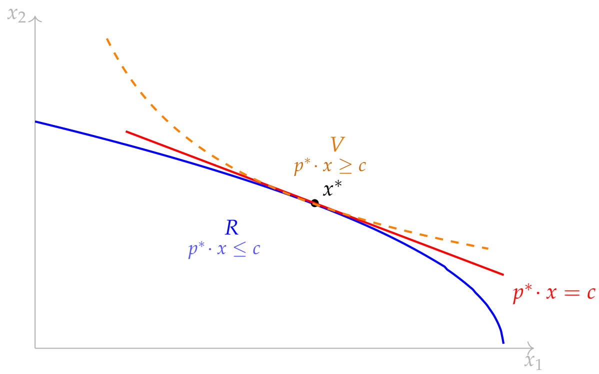
$$p^* \cdot x_i^* \;=\; p^* \cdot \omega_i \;+\; \sum_j \theta_{ij}\, p^* \cdot y_j^* \;+\; T_i,$$

so $x_i^*$ lies on $i$'s budget line. If some affordable $x_i'$ satisfied $x_i' \succ_i x_i^*$ for all $i$ (and at least one strict), then $\sum_i x_i' \in V$ and, using feasibility of $\sum_j y_j^*$, we would have $\sum_i x_i' \in R$, contradicting $R \cap V = \varnothing$ and separation. Hence each $x_i^*$ is utility-maximizing.

**(iii) Market clearing.** By feasibility of $(x^*, y^*)$, $\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*$. Summing the budget equalities yields $\sum_i T_i = 0$. □

Implementation recipe: (1) support the PE frontier with $p^*$; (2) set $T_i$ so $x_i^*$ is exactly affordable; (3) profit and utility maximization + feasibility give a WE with transfers.

**Remark** (Why convexity matters)**.** Without convexity, $R$ or $V$ may be nonconvex ⇒ no separating hyperplane. Then some PE points cannot be decentralized by linear prices (SWT can fail).




Aggregate indifference (orange) is tangent to the PPF (blue) and to the supporting price line (red) at $x^*$.

## *Exercises*

- Consider an exchange economy where consumer 1's preferences are described by the utility function $u_1(x_{11}, x_{12}) = (x_{11})^2 + (x_{12})^2$ and consumer 2's preferences are described by the utility function $u_2(x_{21}, x_{22}) = x_{21}$ (not a typo). The initial endowments are $\omega_i = (5, 5)$ for $i = 1, 2$. Find the set of all Pareto efficient allocations and the set of all Walrasian equilibria for this economy.

- Consider an economy that satisfies all the assumptions of the First Welfare Theorem, but in which the government has decided to tax all profits at a rate of 10%. The tax revenue is distributed equally among all consumers. Does the proof of the First Welfare Theorem still go through?

- Consider an economy that satisfies all the assumptions of the First Welfare Theorem, but in which the government has decided to tax all profits that exceed a certain level $\bar{\pi}$ at 100%, leaving profits that are below $\bar{\pi}$ untaxed. The tax revenue is distributed equally among all consumers. Does the proof of the First Welfare Theorem still go through?

- Find all Pareto-efficient allocations in the following exchange economy. There are 2 consumers $i = 1, 2$. Both consumers have consumption set $X_i = \mathbb{R}^2_+$. For consumer 1 goods 1 and 2 are perfect substitutes: $u_1(x_{11}, x_{12}) = x_{11} + x_{12}$. Consumer 2 has a Cobb–Douglas utility function: $u_2(x_{21}, x_{22}) = x_{21} \cdot x_{22}$. Consumer 1's initial endowment is $(2, 1)$, and consumer 2's initial endowment is $(4, 5)$. For every Pareto-efficient allocation find a price vector and a vector of lump-sum transfers such that this allocation becomes a Walrasian equilibrium.

- Consider an economy with two consumers whose consumption sets are $X_i = \mathbb{R}^3_+$. The first good is leisure, and the second and third goods are consumption goods. Each consumer has an initial endowment of 5 units of leisure, but they have zero endowment of the consumption goods. Consumers don't value leisure. Consumer 1's preferences are represented by

  $$u_1(x_{11}, x_{12}, x_{13}) = (x_{12})^{0.3}(x_{13})^{0.7}.$$

  Consumer 2's preferences are represented by

  $$u_2(x_{21}, x_{22}, x_{23}) = (x_{22})^{0.5}(x_{23})^{0.5}.$$

  There are two firms. If firm 1 uses $y_{11}$ units of leisure, it can produce at most $\sqrt{|y_{11}|}$ units of good 2. Thus, its production set is

  $$Y_1 = \{(y_{11}, y_{12}, y_{13}) \mid y_{11} \leq 0,\ 0 \leq y_{12} \leq \sqrt{|y_{11}|},\ y_{13} = 0\}.$$

If firm 2 uses $y_{21}$ units of leisure, it can produce at most $2\sqrt{|y_{21}|}$ units of good 3. Thus, its production set is

$$Y_2 = \{(y_{21}, y_{22}, y_{23}) \mid y_{21} \leq 0,\ y_{22} = 0,\ 0 \leq y_{23} \leq 2\sqrt{|y_{21}|}\}.$$

Both consumers own one half of each firm.

(a) Find all Walrasian equilibria of this economy.

(b) Confirm that the Walrasian equilibria satisfy the following necessary conditions for Pareto efficiency: (1) the two consumers' marginal rates of substitution between goods 2 and 3 are equal; and (2) the firms equalize the marginal rate of transformation of labor for the two final goods with the consumers' marginal rate of substitution between goods 2 and 3.

(c) Explain intuitively why the two conditions in part (b) are necessary for Pareto efficiency.

- In the example of the previous question, suppose that firm 1 had constant returns to scale:

$$Y_1 = \{(y_{11}, y_{12}, y_{13}) \mid y_{11} \leq 0,\ 0 \leq y_{12} \leq |y_{11}|,\ y_{13} = 0\}.$$

Everything else remains unchanged. Find all Walrasian equilibria.

- Find all Pareto-efficient allocations in the following exchange economy. There are 2 consumers $i = 1, 2$. Both consumers have consumption set $X_i = \mathbb{R}^2_+$. For consumer 1, two units of good 1 and one unit of good 2 are perfect complements: $u_1(x_{11}, x_{12}) = \min\{x_{11}, 2x_{12}\}$ (make sure you understand why the utility formula has the meaning that I claimed it has). Consumer 2 has a Cobb–Douglas utility function: $u_2(x_{21}, x_{22}) = x_{21} \cdot x_{22}$. Consumer 1's initial endowment is $(2, 1)$, and consumer 2's initial endowment is $(6, 3)$. For every Pareto-efficient allocation find a price vector and a vector of lump-sum transfers that add up to zero such that this allocation becomes a Walrasian equilibrium.

# *Class 6: Proof of the Second Welfare Theorem*

Goal today: finish the *proof* of the Second Welfare Theorem (SWT) and then open the door to ranking PE allocations via Social Choice — ending with Arrow's impossibility and a note on the Borda count.

## *From PE to WE via transfers*

Suppose $(x^*, y^*)$ is a Pareto efficient allocation. We want to decentralize it as a Walrasian equilibrium with transfers. The strategy:

1. Find a supporting price vector $p^*$. (Separating/supporting hyperplane for the feasible set at $\sum_i x_i^*$.)
2. With $p^*$ in hand, define lump-sum transfers that make each $x_i^*$ exactly affordable.

**Definition** (Transfers that implement $(x^*, y^*)$)**.** For ownership shares $\{\theta_{ij}\}_{i,j}$ with $\sum_i \theta_{ij} = 1$ for each firm $j$, set for each consumer $i$:

$$T_i \;=\; p^* \cdot x_i^* \;-\; p^* \cdot \omega_i \;-\; \sum_{j=1}^{J} \theta_{ij}\, p^* \cdot y_j^*.$$

**Lemma** (Budget balance)**.** *With the $T_i$ above, $\sum_{i=1}^{I} T_i = 0$.*

*Proof.* Using $\sum_i \theta_{ij} = 1$ and feasibility $\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*$,

$$\sum_i T_i = p^* \cdot \sum_i x_i^* - p^* \cdot \sum_i \omega_i - \sum_j p^* \cdot y_j^* = 0.$$

□

**Proposition** (WE with transfers)**.** *At prices $p^*$ and transfers $T = (T_i)_i$ above, $(p^*, x^*, y^*)$ is a Walrasian equilibrium with transfers.*

*Sketch.* (i) Each $y_j^*$ is profit-maximizing at $p^*$ (supporting hyperplane at the aggregate feasibility point). (ii) By construction $x_i^*$ lies on $i$'s budget line: $p^* \cdot x_i^* = p^* \cdot \omega_i + \sum_j \theta_{ij}\, p^* \cdot y_j^* + T_i$. If there were an affordable strict improvement for everyone, it would contradict separation. (iii) Feasibility holds by construction. □

## *Geometry: the two convex sets*

*R:* Aggregate feasible consumption vectors: $R = \{\, \sum_i \omega_i + \sum_j y_j : y_j \in Y_j \,\}$ (production + endowment).

$V$: Aggregates that make everyone weakly better than $x^*$ and someone strictly better.

Both sets are convex under the standing convexity assumptions (and Minkowski sums preserve convexity). At PE, $R \cap V = \emptyset$, and $x^*$ lies on the boundary supported by the price hyperplane $p^* \cdot x = c$.

### *Why convexity matters*

Convexity of preferences/technologies ensures $R$ and $V$ are convex and disjoint, so a *supporting* (separating) hyperplane exists through $\sum_i x_i^*$ with normal $p^* \gg 0$. This is the price system that decentralizes $(x^*, y^*)$ once we add lump-sum $T$.

## *Can we* rank *Pareto-efficient allocations? Social Choice*

Motivation: PE typically gives a *set* (frontier/contract curve). To select among PE points we need an ordering over alternatives built from individual preferences.

### *Profiles and aggregation*

Let the set of social alternatives be $A = \{a, b, c, \dots\}$. There are $N$ agents. Agent $i$ has a complete and transitive preference $R_i$ over $A$. A *profile* is $R = (R_1, \dots, R_N)$. A *social welfare functional* (a.k.a. preference aggregation rule) is a map

$$f : \mathcal{R}^N \to \mathcal{R},$$

which assigns to each profile a complete and transitive social ordering $R = f(R_1, \dots, R_N)$. We want $R$ to *represent* unanimous comparisons: if $a\, R_i\, b$ for all $i$, then $a\, R\, b$.

### *Arrow's axioms*

Extra: There is quite a good video of this from Veritasium here

***Pareto Efficiency*** If $x\, R_i\, y$ for all $i$, then $x\, R\, y$.

***No Dictatorship*** There is no individual $i$ such that for all profiles $R$ and all $x, y \in A$, if $x\, R_i\, y$ then $x\, R\, y$.

***Independence of Irrelevant Alternatives (IIA)*** For any $x, y \in A$, the social ranking of $x$ vs. $y$ depends only on the agents' rankings of $x$ vs. $y$ (not on preferences over other options).

**Theorem** (Arrow's Impossibility Theorem)**.** *If $|A| \geq 3$, there is no aggregation rule $f$ that satisfies Pareto Efficiency, No Dictatorship, and IIA simultaneously.*

**Remark** (Borda count)**.** Named example rule to keep in mind for later: the *Borda count*. (We won't analyze it here.)

*IIA restated*

Let $x, y \in A$. Consider two profiles $R = (R_1, \ldots, R_N)$ and $\hat{R} = (\hat{R}_1, \ldots, \hat{R}_N)$ that agree on the pair $(x, y)$ for every agent:

$$x\, R_i\, y \iff x\, \hat{R}_i\, y \qquad (i = 1, \ldots, N).$$

Then IIA requires that the social ranking of $x$ vs. $y$ is identical under $R$ and under $\hat{R}$:

IIA says: only the agents' pairwise rankings of $(x, y)$ may matter for the social ranking of $(x, y)$.

$$x\, f(R)\, y \text{ and } x\, f(\hat{R})\, y \quad \text{or} \quad y\, f(R)\, x \text{ and } y\, f(\hat{R})\, x.$$

Intuition: if nobody changes how they compare $x$ and $y$, society cannot change how it compares $x$ and $y$, regardless of movements over other options.

*Majority voting and the Condorcet cycle*

**Definition** (Simple majority relation)**.** For a profile $R$, define $a\, M(R)\, b$ iff $|\{\, i : a\, P_i\, b \,\}| \geq N/2$.

**Example** (Condorcet cycle)**.** With three agents and three alternatives $A = \{a, b, c\}$, let

| | $i$=1 | $i$=2 | $i$=3 |
|---|---|---|---|
| 1st | $a$ | $b$ | $c$ |
| 2nd | $b$ | $c$ | $a$ |
| 3rd | $c$ | $a$ | $b$ |

Then a majority prefers $a$ to $b$, a majority prefers $b$ to $c$, and a majority prefers $c$ to $a$. Hence $M(R)$ is *intransitive* (a **Condorcet cycle**).

**Remark.** Majority rule can violate transitivity, so it need not produce a complete and transitive social ordering. This dovetails with Arrow's theorem: with $|A| \geq 3$, Pareto + IIA + No Dictator cannot all hold for any aggregation rule.

*Takeaway (addendum):*

- IIA: two profiles that agree on each agent's comparison of $(x, y)$ must induce the same social ranking of $(x, y)$.
- Pairwise majority can cycle: $a \succ b$, $b \succ c$, $c \succ a$ by majority.

# *Class 7: Arrow, Generalized Majority Rules, and Escaping Impossibility*

Roadmap. (i) Restate Arrow's theorem (axioms and statement). (ii) Formalize *generalized majority voting* via winning coalitions and show when it degenerates into dictatorship or cycles. (iii) Explain how strengthening the domain (VNM or quasi–linear) enables cardinal aggregation (Harsanyi; surplus maximization).

## *Setup: profiles and social aggregation*

Let $A$ be a finite, nonempty set of social alternatives, $|A| \geq 3$ unless noted. There are $N = \{1, \ldots, n\}$ agents. A *profile* is $R = (R_1, \ldots, R_n)$, where each $R_i$ is a complete and transitive preference on $A$; write $P_i$ and $I_i$ for strict and indifference parts. A *social welfare functional* (SWF) is a map

$$f : \mathcal{R}^n \to \mathcal{R}, \qquad R \mapsto R^S = f(R),$$

which associates to each profile a complete and transitive *social* ordering $R^S$ on $A$.

## *Arrow's axioms and impossibility*

***Unrestricted Domain (UD).*** $\mathcal{R}$ is the set of all complete and transitive relations on $A$; any profile $R \in \mathcal{R}^n$ is admissible.

***Weak Pareto (WP).*** If $x\, R_i\, y$ for all $i$, then $x\, R^S\, y$.

***Independence of Irrelevant Alternatives (IIA).*** For any $x, y \in A$, if two profiles $R, \hat{R}$ satisfy $x\, R_i\, y \iff x\, \hat{R}_i\, y$ for every $i$, then $x\, R^S\, y \iff x\, \hat{R}^S\, y$.

***No Dictatorship (ND).*** There is no $i$ such that $x\, R_i\, y \Rightarrow x\, R^S\, y$ for all $x, y$ and all profiles.

**Theorem** (Arrow's Impossibility)**.** *If $|A| \geq 3$ and UD, WP, IIA, ND hold, then there is no SWF $f$ satisfying all four axioms simultaneously.*

Intuition. UD + IIA let pairwise social comparisons depend only on pairwise individual data, uniformly across profiles. WP forces responsiveness. Together they generate "decisive coalitions" whose structure, under finiteness, collapses to a dictator.

## *Generalized majority voting (GMV)*

### *Definition via winning coalitions*

A *winning coalition system* is a nonempty family $\mathcal{B} \subseteq 2^N$. Given $\mathcal{B}$, define the *GMV social weak preference* $M_{\mathcal{B}}(R)$ by

$$x\, M_{\mathcal{B}}(R)\, y \quad \iff \quad \{\, i \in N : \; x\, R_i\, y \,\} \in \mathcal{B}.$$

Interpretation: $x$ is socially at least as good as $y$ if the set of individuals who (weakly) prefer $x$ to $y$ is "winning." Let $k_{\min} := \min\{\, |S| : S \in \mathcal{B} \,\}$ (minimal winning size).

### *Regularity conditions on $\mathcal{B}$*

For $M_{\mathcal{B}}(R)$ to be a sensible social ordering across all profiles, the following structural conditions are natural:

1. *Monotonicity (upward-closed).* If $S \in \mathcal{B}$ and $S \subseteq T \subseteq N$, then $T \in \mathcal{B}$.
2. *Properness.* For every $S \subseteq N$, exactly one of $S$ or $N \setminus S$ belongs to $\mathcal{B}$.
3. *Finite intersection property.* If $S, T \in \mathcal{B}$, then $S \cap T \in \mathcal{B}$.

A family $\mathcal{B}$ satisfying (i)–(iii) is an *ultrafilter* on $N$.

**Proposition** (GMV and transitivity)**.** *If $\mathcal{B}$ fails (iii), then there exist profiles for which $M_{\mathcal{B}}(R)$ is intransitive. If $\mathcal{B}$ is an ultrafilter, then for every profile $R$ the relation $M_{\mathcal{B}}(R)$ is complete and transitive.*

**Proposition** (Ultrafilters on finite electorates)**.** *If $N$ is finite and $\mathcal{B}$ is an ultrafilter, then $\mathcal{B}$ is* principal*: there exists $i^* \in N$ with $\{i^*\} \in \mathcal{B}$. Consequently $M_{\mathcal{B}}$ is* dictatorial *with dictator $i^*$.*

**Corollary** (Minimal winning size and pathologies)**.** *Let*

$$k_{\min} := \min\{\, |S| : S \in \mathcal{B} \,\}$$

- *If $k_{\min} = 1$, the rule is dictatorial.*
- *If $k_{\min} \leq 2$ and the rule is not dictatorial, there exists a profile with a Condorcet cycle $a \succ b \succ c \succ a$ under $M_{\mathcal{B}}$.*

Takeaway. Requiring a pairwise rule to be transitive for *all* profiles forces $\mathcal{B}$ to be an ultrafilter; on finite $N$ this is a dictatorship. Majority rule (simple $\frac{n}{2}$-threshold) is not an ultrafilter, hence can cycle.

## *Beyond Arrow: strengthening the domain*

### *Cardinal utilities (VNM) and Harsanyi*

Let $\Delta(A)$ denote lotteries over $A$. Suppose each $i$ has VNM preferences on $\Delta(A)$ with utility $u_i$. Normalize interpersonally by

$$\min_{a \in A} u_i(a) = 0, \qquad \max_{a \in A} u_i(a) = 1.$$

A classical aggregation result (Harsanyi) shows that, under Pareto-type and symmetry/continuity axioms on social preferences over lotteries, there exist weights $(\lambda_i)_i$ with $\lambda_i > 0$ such that the social ranking is represented by

$$a \mapsto \sum_{i=1}^{n} \lambda_i u_i(a).$$

*Comment.* This abandons UD and IIA: it uses a *restricted, cardinal domain* (VNM) and *interpersonal comparability* via normalization.

## *Quasi–linear preferences and willingness to pay*

### *Axioms and representation*

Each agent consumes $(x, m) \in X_i \times \mathbb{R}$, where $x$ denotes non-numeraire goods and $m$ is money (numeraire; can be negative). Assume:

1. **Monotonicity in money:** $(x, m) \succeq_i (x, m') \iff m \geq m'$.
2. **Translation invariance in money:** $(x, m) \succeq_i (x', m') \iff (x, m + t) \succeq_i (x', m' + t)$ for all $t \in \mathbb{R}$ ("WTP does not depend on wealth level").
3. **Richness:** For all $x, x'$ there exist $m, m'$ with $(x, m) \sim_i (x', m')$ ("anything can be reached with money").

**Proposition** (Quasi–linear representation)**.** *Under (a)–(c) there exists a function $u_i : X_i \to \mathbb{R}$ such that preferences admit the representation*

$$(x, m) \mapsto u_i(x) + m.$$

*The function $u_i(x)$ is defined up to an additive constant; money provides the cardinal numeraire.*

**Definition** (Money–metric utility)**.** Fix a reference bundle of money $\bar{m} = 0$. The money–metric welfare of $x$ for agent $i$ is

$$M_i(x) := \sup\{\, m : (x, m) \succeq_i (x', 0) \text{ for some fixed reference } x' \,\}.$$

Under quasi–linearity and the normalization above, $M_i(x) = u_i(x)$ up to a common additive constant.

### *Efficiency with transferable money*

Let the economy be feasible in $(x, m)$ with $\sum_i m_i$ fixed by aggregate resources. Because money is perfectly redistributable, the utility possibility set is a translate of the $u$–possibility set.

**Theorem** (Surplus maximization $\Leftrightarrow$ Pareto efficiency). *An allocation* $(x^*, m^*)$ *is Pareto efficient if and only if* $x^*$ *maximizes*

$$\sum_{i=1}^{n} M_i(x_i) \quad \textit{over all feasible } x,$$

*and* $m^*$ *is any vector of transfers that clears the money resource. Equivalently, with the normalization* $M_i \equiv u_i$*, efficiency is equivalent to maximizing* $\sum_i u_i(x_i)$ *subject to feasibility.*

*Sketch.* If $x^*$ maximizes $\sum_i M_i(x_i)$, any feasible $x$ has $\sum_i M_i(x_i) \leq \sum_i M_i(x_i^*)$; because money is numeraire, we can redistribute $\{m_i\}$ to keep everyone at least as well off as under $x^*$ only if the sum does not increase—hence $x^*$ is PE. Conversely, if $x^*$ were not maximizing the sum, some feasible $x$ would give a strictly higher $\sum_i M_i(x_i)$; then, by distributing the strictly positive surplus in money, everyone can be made weakly better off and at least one strictly better off, contradicting PE. □

Graphically: with quasi–linear utility, the utility–possibility frontier has slope $-1$ in $(u_1, u_2)$ space because one more unit of $u_1$ can be financed by one less unit of $u_2$ via money transfers. PE picks the point that maximizes $u_1 + \cdots + u_n$ subject to feasibility.

## *Key takeaways*

- **Arrow.** UD + IIA + WP force the structure of pairwise aggregation into winning–coalition systems; transitivity for all profiles $\Rightarrow$ ultrafilter $\Rightarrow$ dictatorship on finite electorates.
- **Cycles vs. dictatorship.** If $k_{\min} \leq 2$ and the rule is *non-dictatorial*, there exist profiles with a Condorcet cycle; if $k_{\min} = 1$, the rule is dictatorial.
- **Escaping impossibility.** Restricting the domain to VNM (lotteries) and adopting interpersonal comparability yields utilitarian aggregation (Harsanyi).
- **Quasi–linear world.** Money is a perfect numeraire; efficiency is equivalent to maximizing the sum of (money–metric) utilities and then redistributing money via transfers.

## *Exercises*

- Consider two Edgeworth box exchange economies (each with two goods and two consumers), and assume that initially they are two entirely separate economies where trade takes place only within each economy, but not between economies. Then, in a second step, assume that the economies are integrated, so that they are now one exchange economy with four consumers. Assume that good 1 in exchange economy 1 is identical to good 1 in exchange economy 2, and that good 2 in exchange economy 1 is identical to good 2 in exchange economy 2, so that the integrated economy has just two markets. Assume that all markets of all exchange economies considered in this question are perfectly competitive.

  (a) Construct an example in which all four consumers are better off after the integration of the two economies than they were before.

  (b) Construct an example in which at least one consumer is worse off after the integration of the two economies than she was before.

  (c) Use the second welfare theorem to argue that, when lump sum transfers and taxes can be used, it is always possible to make *all* consumers better off (or at least not worse off) after the economies are integrated than they were before the economies were integrated.

- Suppose a society has to rank four alternatives: $A = \{a, b, c, d\}$. There are 3 individuals. Each individual has a complete and transitive preference over these alternatives. Society's welfare ordering is equal to the preference of individuals 1 and 2 as long as individuals 1 and 2 have the same preference. Otherwise, society's welfare ordering is equal to the preference of individual 3. Which of the following axioms does this system satisfy and which does it violate: completeness and transitivity of society's ordering, Pareto axiom, independence of irrelevant alternatives, non dictatorship? If your answer is that a property is violated, prove this by giving a counterexample.

- Consider a society that consists of three agents: $i = 1, 2, 3$. Society has to choose one policy from the set $\{a, b, c\}$ and society also has \$50 which can be divided in arbitrary ways among agents. Thus, an alternative is a 4–tuple such as: $(a, 10, 20, 20)$ which means that the policy $a$ is implemented, agent 1 receives \$10 and agents 2 and 3 receive \$20. In general, the set of alternatives is

$$\{(x, m_1, m_2, m_3) \mid x \in \{a, b, c\},\ m_1, m_2, m_3 \in \mathbb{N}_0,\ m_1 + m_2 + m_3 = 50\}.$$

Notice that we have restricted the monetary payments to the non–negative integers, and that we have assumed in this formulation that all \$50 have to be given away.

Agents have ordinal preferences over alternatives. Let the set of preferences that agent $i$ may possibly have be denoted by $\mathcal{R}_i$. This set consists of the set of all ordinal preferences that can be represented by a quasi–linear utility function of the form:

$$u_i(x, m_1, m_2, m_3) = v_i(x) + m_i$$

where $v_i$ can be a function with domain $\{a, b, c\}$ and co–domain $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Note that for each ordinal preference $\mathcal{R}_i$ there is exactly one numerical representation of the form described in the above equation. There cannot be multiple such representations.

Suppose we construct society's welfare ordering by ordering alternatives in accordance with the sum of utilities:

$$\sum_{i \in \{1,2,3\}} u_i(x, m_1, m_2, m_3).$$

Which of Arrow's axioms, if any, does this method of aggregating preferences satisfy and which, if any, does it violate on this restricted domain?

- Consider an exchange economy with two agents and two goods. Each agent's consumption set is $\mathbb{R}^2_+$. The economy has a total initial endowment of 50 units of each of the two goods. Each agent $i$'s preferences are represented by the utility function

$$u_i(x_{i1}, x_{i2}) = \sqrt{x_{i1}} + x_{i2}.$$

Describe the set of all feasible allocations that are Pareto efficient, and do *not* maximize the sum of utilities $\sum_{i=1}^{2} u_i(x_{i1}, x_{i2})$.

# *Class 8: Restricted Domains, Quasi–Linear Economies, and Partial–Equilibrium Foundations*

## *Restricted domains and why Arrow no longer bites*

Roadmap. (i) How restricting the *domain* weakens Arrow (single–peaked, VNM). (ii) Quasi–linear representation and money–metric utility. (iii) Planner vs. competitive equilibrium in a quasi–linear exchange–production economy. (iv) Partial–equilibrium demand/supply as welfare maximization.

We recall Arrow's impossibility (UD, WP, IIA, ND) and ask which restrictions on the *domain* of admissible individual preferences open doors.

### *Single–peaked preferences on a line*

Let $(A, \lhd)$ be a linearly ordered set of alternatives. Preference $R_i$ is *single–peaked* (w.r.t. $\lhd$) if there exists a *peak* $p_i \in A$ such that moving away from $p_i$ in either direction (in $\lhd$) never improves the ranking.

**Proposition** (Majority transitivity on a single–peaked domain)**.** *If all $R_i$ are single–peaked w.r.t. the same $\lhd$, then the simple majority relation is transitive and admits a Condorcet winner (the* median peak*).*

**Remark.** This does not contradict Arrow: UD is replaced by a *restricted* domain; moreover, majority rule on this domain generally fails IIA. The lesson is: *structure the domain* to avoid cycles.

### *Lotteries and VNM utilities (Harsanyi's route)*

Let $\Delta(A)$ denote lotteries over $A$. If each agent satisfies VNM axioms, preferences are represented by cardinal utilities $u_i$ that are *affine* in probabilities. With an interpersonal normalization (e.g., $\min_a u_i(a) = 0$, $\max_a u_i(a) = 1$), social preferences over lotteries can be represented by a weighted utilitarian aggregator $\sum_i \lambda_i u_i$. This again escapes Arrow by restricting the domain and abandoning IIA.

## *Quasi–linear preferences and money–metric utility*

### *Axioms and representation*

Each agent consumes $(x, m) \in X_i \times \mathbb{R}$, where $m$ is numeraire money. Assume:

1. **Monotonicity in money:** $(x, m) \succeq_i (x, m') \iff m \geq m'$.
2. **Translation invariance in money:** $(x, m) \succeq_i (x', m') \iff (x, m + t) \succeq_i (x', m' + t)$ for all $t \in \mathbb{R}$.
3. **Richness:** For all $x, x'$ there exist $m, m'$ with $(x, m) \sim_i (x', m')$.

Then there exists $u_i : X_i \to \mathbb{R}$ such that preferences admit the *quasi–linear* representation

$$(x, m) \mapsto u_i(x) + m, \qquad \text{unique up to an additive constant.}$$

**Definition** (Money–metric utility)**.** Fix a reference money level $\bar{m} = 0$. The money–metric welfare of $x$ for agent $i$ is $M_i(x) := \sup\{m : (x, m) \succeq_i (x', 0)\}$ for some fixed reference $x'$. Under quasi–linearity and the above normalization, $M_i(x) = u_i(x)$ up to a common additive constant.

## *A quasi–linear exchange–production economy*

### *Environment*

Two goods: $\ell = 1$ is money (price normalized to 1), $\ell = 2$ is a consumption good with price $p$. Consumers $i = 1, \ldots, I$ have utilities

$$U_i(m_i, x_i) = m_i + \phi_i(x_i), \qquad \phi_i' > 0,\ \phi_i'' < 0,\ x_i \geq 0.$$

Firms $j = 1, \ldots, J$ produce the consumption good with convex costs $C_j(q_j)$: technology $y_j = (-z_j, q_j)$ with $z_j \geq C_j(q_j)$ and $q_j \geq 0$.

*Feasibility.* Let consumers hold initial money endowments $\omega_i$ and no units of the consumption good. Then

$$\sum_{i=1}^{I} x_i \leq \sum_{j=1}^{J} q_j, \qquad \sum_{i=1}^{I} m_i + \sum_{j=1}^{J} z_j = \sum_{i=1}^{I} \omega_i, \qquad z_j \geq C_j(q_j).$$

Eliminating money via $z_j \geq C_j(q_j)$, feasibility is equivalent to

$$\sum_{i=1}^{I} m_i \leq \sum_{i=1}^{I} \omega_i - \sum_{j=1}^{J} C_j(q_j), \qquad \sum_i x_i \leq \sum_j q_j.$$

### *Planner's problem and characterization of efficiency*

Consider the planner program (money is freely redistributable):

$$\max_{\{x_i \geq 0\},\{q_j \geq 0\}} \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} C_j(q_j) \quad \text{s.t.} \quad \sum_i x_i \leq \sum_j q_j. \tag{1}$$

**Theorem** (PE ⇔ surplus maximization)**.** *An allocation* $(x^*, m^*; q^*, z^*)$ *is Pareto efficient if and only if* $(x^*, q^*)$ *solves* (1)*. Transfers* $m^*$ *then implement any point on the money dimension consistent with feasibility.*

*Sketch.* Quasi–linearity implies that money is a perfect numeraire; hence only $\{x_i\}, \{q_j\}$ matter for Pareto rankings. If $(x^*, q^*)$ solves (1) and one could find a feasible $(x, q)$ with a higher objective, distributing the surplus in money would make everyone weakly better off and someone strictly better off: contradiction. The converse is analogous. □

### *Competitive equilibrium and partial–equilibrium FOCs*

Given prices $(1, p)$ and individual incomes $B_i$ (from endowments/ profits/transfers), each consumer solves

$$\max_{x_i \geq 0} \ \phi_i(x_i) - p x_i \quad (\text{plus a constant } B_i),$$

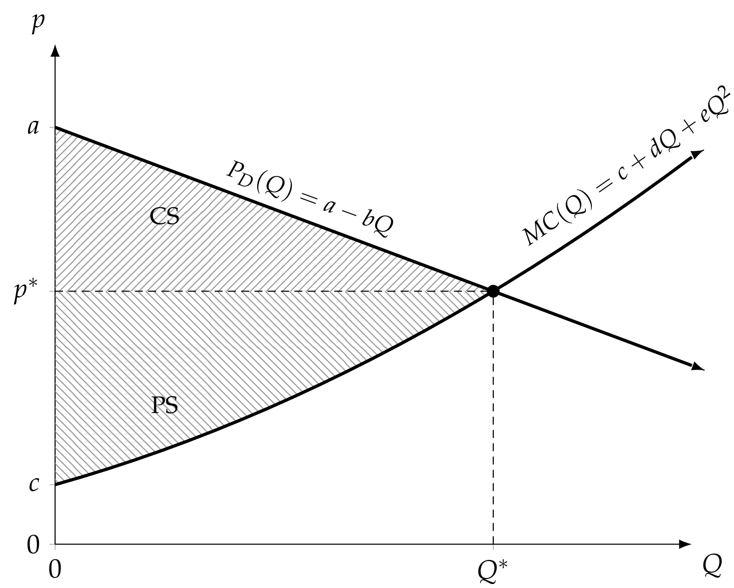
so the KKT conditions are $\phi_i'(x_i) \leq p$ with equality when $x_i > 0$. Each firm solves $\max_{q_j \geq 0} \ p q_j - C_j(q_j)$, yielding $C_j'(q_j) \geq p$ with equality if $q_j > 0$. Market clearing requires $\sum_i x_i = \sum_j q_j$.

**Proposition** (Welfare theorems in the quasi–linear model)**.** *Any competitive equilibrium* $(p^*, x^*, q^*)$ *solves the planner problem* (1)*, and conversely any solution to* (1) *can be decentralized with transfers as a competitive equilibrium at price* $p^*$*.*

*Geometric reading (partial equilibrium).* Ordering individual marginal WTPs $\phi_i'$ from highest to lowest yields an *aggregate inverse demand* $P_D(Q)$. With aggregate marginal cost $MC(Q)$ (from the convex envelope of $\{C_j'\}$), the planner objective equals

$$\int_0^Q P_D(q)\, dq \; - \; \int_0^Q MC(q)\, dq,$$

whose maximizer satisfies $P_D(Q) = MC(Q)$. Thus, the usual supply–demand cross is literally a welfare program under quasi–linearity.


p
a
CS
$P_D(Q) = a - bQ$
$MC(Q) = c + dQ + eQ^2$
$p^*$
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# *Class 9: Existence of at least one Walrasian Equilibrium*

## *A numerical non-existence example*

*Environment.* Two goods $\ell = 1, 2$ and two consumers $i = 1, 2$. Consumption sets $X_i = \mathbb{R}^2_+$. Endowments:

$$\omega_1 = (3, 2), \qquad \omega_2 = (2, 3).$$

Normalize $p_1 = 1$ and write the price vector as $p = (1, p_2) \in \mathbb{R}^2_{++}$.
**Preferences.**

- Consumer 1 has strictly monotone, convex preferences that induce *corner* choices except at the tie $p_2 = 1$. Concretely, her indirect choice behaves like a strict-selection version of perfect substitutes: if one good is strictly cheaper, she spends the entire budget on that good; at $p_2 = 1$ she selects an endpoint of the budget segment (details below). This selection rule will be pivotal.
- Consumer 2 has Cobb–Douglas utility $u_2(x_{21}, x_{22}) = x_{21}\, x_{22}$.

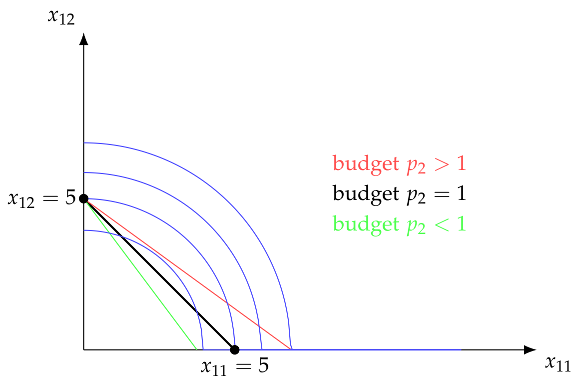
*Individual demands.* Let $m_i(p) = p \cdot \omega_i$ denote income.

*Consumer 1* Her spending rule is:

$$x_{12}(p) \;=\; \begin{cases} 0, & \text{if } p_2 > 1, \\ \{0,\, 5\}, & \text{if } p_2 = 1, \\ \dfrac{m_1(p)}{p_2} = \dfrac{3 + 2p_2}{p_2}, & \text{if } p_2 < 1, \end{cases} \qquad m_1(p) = 3 + 2p_2.$$

*Intuition.* When $p_2 < 1$ good 2 is strictly cheaper, so she spends all income on good 2; when $p_2 > 1$ she buys only good 1; at the knife-edge $p_2 = 1$ she (by strict selection) chooses an endpoint of the budget line, delivering the two-point set $\{0, 5\}$ for $x_{12}$.

**Remark.** If Consumer 1 had *exact* perfect-substitutes utility and we kept the full argmax *correspondence*, then at $p_2 = 1$ her demand for good 2 would be the whole interval $[0, 5]$. That convex, interval-valued demand at the tie will matter for existence.




Consumer 1 (non–convex preferences): at $p_2 = 1$ the maximizers are the two corners.

*Consumer 2* Cobb–Douglas with $u_2 = x_{21}x_{22}$ implies constant expenditure shares. With

$$m_2(p) = p \cdot \omega_2 = 2 + 3p_2,$$

the Marshallian demand for good 2 is

$$x_{22}(p) = \frac{1}{2}\frac{m_2(p)}{p_2} = \frac{2+3p_2}{2p_2}.$$

## *Market for good 2*

Aggregate endowment (supply) of good 2 is fixed at

$$\omega_1^2 + \omega_2^2 = 2 + 3 = 5.$$

Aggregate *demand* for good 2 is $x_{12}(p) + x_{22}(p)$, which is piecewise:
*Case $p_2 < 1$.*

$$x_{12}(p) + x_{22}(p) = \frac{3+2p_2}{p_2} + \frac{2+3p_2}{2p_2} = \frac{4}{p_2} + \frac{7}{2}.$$

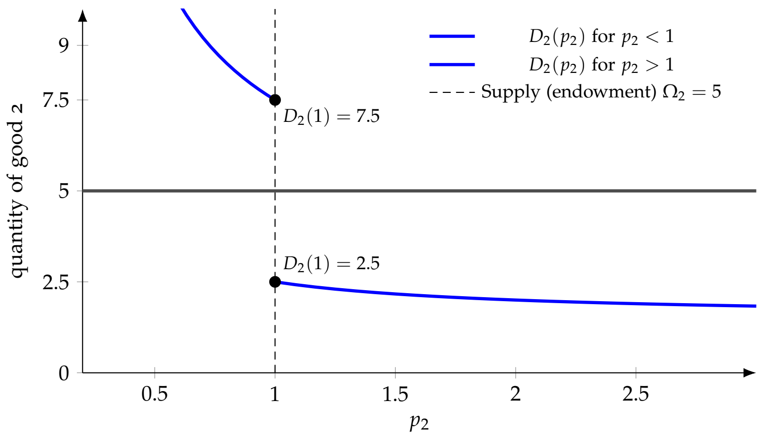
Market clearing $x_{12} + x_{22} = 5$ would require $\frac{4}{p_2} + \frac{7}{2} = 5$, i.e. $\frac{4}{p_2} = \frac{3}{2}$, so $p_2 = \frac{8}{3} > 1$, contradicting $p_2 < 1$.
*Case $p_2 > 1$.*

$$x_{12}(p) + x_{22}(p) = 0 + \left(\frac{1}{p_2} + \frac{3}{2}\right).$$

Market clearing requires $\frac{1}{p_2} + \frac{3}{2} = 5$, i.e. $\frac{1}{p_2} = \frac{7}{2}$, so $p_2 = \frac{2}{7} < 1$, contradicting $p_2 > 1$.
*Case $p_2 = 1$.*

$$x_{22}(p) = \frac{2+3}{2} = 2.5, \qquad x_{12}(p) \in \{0, 5\}.$$

Clearing the good 2 market requires $x_{12} = 5 - 2.5 = 2.5$, but the selection delivers only $\{0, 5\}$. Hence, no market-clearing choice exists at $p_2 = 1$.




Good 2 market: aggregate demand never hits the fixed supply at 5.

**Remark.** At $p_2 = 1$, Consumer 1's selection $\{0, 5\}$ is discontinuous: it jumps from full demand of good 2 to none, omitting any interior choice. If instead her demand correspondence were the convex set $[0, 5]$, the map would be upper hemicontinuous (by Berge's Maximum Theorem), preserving convexity and ensuring an equilibrium. The restricted selection $\{0, 5\}$ breaks u.h.c. and destroys existence.

**Conclusion.** There is *no* $p_2 > 0$ for which the market for good 2 clears. Therefore, a Walrasian equilibrium does not exist in this economy (with Consumer 1's strict-selection rule).

## *Equilibrium as a fixed point problem*

In equilibrium, aggregate demand equals aggregate supply:

$$\text{demand}(p) = \text{supply}(p) \quad \iff \quad z(p) = \text{demand}(p) - \text{supply}(p) = 0,$$

where $z(p)$ denotes the *excess demand function*[3]. A Walrasian equilibrium (W.E.) corresponds to any price vector $p$ satisfying $z(p) = 0$.

*Defining the domain.* Let $\Delta$ denote the *price simplex*,

$$\Delta = \{(p_1, p_2, \ldots, p_L) \in \mathbb{R}^L_+ \mid p_1 + p_2 + \cdots + p_L = 1\}.$$

We define $z : \Delta \to \mathbb{R}^L$ as a single-valued excess demand function (for simplicity we do not allow correspondences here).[4]

**Proposition.** *If z satisfies:*

1. *z is continuous,*

2. *z satisfies Walras' Law: $p \cdot z(p) = 0$ for all $p \in \Delta$,*

[3] The term *excess of demand* has a simple meaning: it measures, for each good, the difference between the desired consumption at prices $p$ and the available resources. When $z_\ell(p) > 0$, the market demands more than what is available (pressure for the price to rise). When $z_\ell(p) < 0$, there is excess supply (pressure for the price to fall). Hence equilibrium is the situation in which all these pressures disappear.

[4] If we used demand correspondences then excess demand would itself be a *correspondence* $Z(p)$, and proof would rely on **Kakutani's** fixed–point. We would need: (i) each individual demand correspondence to be nonempty, compact, *convex*, and *u.h.c.*; (ii) Walras' Law $p \cdot z = 0$ for all $z \in Z(p)$; (iii) a *boundary condition*: if $p_\ell = 0$, then some $z \in Z(p)$ satisfies $z_\ell > 0$ (to avoid zero–price goods). Under these assumptions, Kakutani ensures $p^*$ and $z^* \in Z(p^*)$ with $z^* = 0$.

3. *if $p_\ell = 0$, then $z_\ell(p) > 0$,*

*then there exists at least one price vector $p^* \in \Delta$ such that $z(p^*) = 0$.*

*Idea of the proof.* The result follows from **Brouwer's Fixed Point Theorem**.

**Remark** (Comment on the alternative approach.)**.** If we used a *contraction mapping theorem* instead of Brouwer, we could guarantee convergence of the iterative price-adjustment process to the fixed point from any starting price. However, that condition is much more restrictive and is not required for the existence proof.

**Theorem** (Brouwer Fixed Point Theorem)**.** *If $X \subseteq \mathbb{R}^n$ is non-empty, convex, and compact, and $f : X \to X$ is continuous, then there exists $x^* \in X$ such that $f(x^*) = x^*$.*

*Constructing a continuous map on the simplex.* Define $f : \Delta \to \Delta$ by

$$f_\ell(p) = \frac{p_\ell + \max\{z_\ell(p), 0\}}{\sum_{k=1}^{L}(p_k + \max\{z_k(p), 0\})}.$$

Intuitively, if a good $\ell$ has positive excess demand, $\max\{z_\ell(p), 0\} > 0$ increases its relative weight in the normalized price vector. If instead there is excess supply ($z_\ell(p) < 0$), the denominator increases while the numerator does not, reducing $f_\ell(p)$.

**Remark.** The idea is to find a fixed point of this transformation of prices: a price vector $p$ such that $f(p) = p$. At that point, the "adjusted" prices already balance excess demand, i.e. $z(p) = 0$.

*Step 1.* By Brouwer's theorem, since $\Delta$ is compact and convex, and $f$ is continuous, there exists at least one fixed point $p^*$ such that $f(p^*) = p^*$.

*Step 2.* Show that any such fixed point must satisfy $z(p^*) = 0$. Suppose $f(p) = p$; then for each good $\ell$,

$$\frac{p_\ell + \max\{z_\ell(p), 0\}}{\sum_{k=1}^{L}(p_k + \max\{z_k(p), 0\})} = p_\ell.$$

Pick some $\ell$ such that $z_\ell(p) \leq 0$ (such a good must exist because by Walras' Law not all $z_\ell(p)$ can be strictly positive). Then

$$\frac{p_\ell}{\sum_{k=1}^{L}(p_k + \max\{z_k(p), 0\})} = p_\ell \quad\Longrightarrow\quad \sum_{k=1}^{L}(p_k + \max\{z_k(p), 0\}) = 1.$$

This is only possible if $\max\{z_k(p), 0\} = 0$ for all $k$, i.e. $z_k(p) \leq 0$ for all goods.

Now, if all goods exhibit excess supply ($z_k(p) \leq 0$), then by Walras' Law:[5]

$$p \cdot z(p) = \sum_{k=1}^{L} p_k z_k(p) < 0,$$

a contradiction. Hence, the only possible case is $z(p) = 0$.

[5] Shouldn't it be $\leq$?

*Conclusion.* All goods with excess supply would have price zero, but we ruled out that possibility by assuming $p_\ell > 0$ for all $\ell$. Therefore, the fixed point $p^*$ satisfies $z(p^*) = 0$, which is a Walrasian equilibrium.

# *Class 10: Basic axioms on the primitives for the existence of Walrasian equilibrium*

Last class we constructed a (single–valued) excess demand function

$$z : \Delta \to \mathbb{R}^L$$

and used it to prove the existence of a Walrasian equilibrium under the following three conditions:

1. $z(p)$ is continuous in $p$,
2. $z(p)$ satisfies Walras' Law: $p \cdot z(p) = 0$ for all $p \in \Delta$,
3. if $p_\ell = 0$, then $z_\ell(p) > 0$.

Any $p \in \Delta$ such that $z(p) = 0$ is then a Walrasian equilibrium price vector.

In this class we go "one level back": we look for assumptions on the *primitives* of the economy (consumption sets, preferences, endowments, technologies) that guarantee these three properties of the excess demand function.

**Theorem.** *If, for every consumer $i = 1, \ldots, I$ and every firm $j = 1, \ldots, J$, the primitives satisfy:*

1. $X_i = [0, m]^L$ *for some large* $m > 0$*, with*

$$m > \max_{\ell=1,\ldots,L} x_\ell \qquad \text{where} \qquad x_\ell \; = \; \sum_{i=1}^{I} \omega_{i\ell} + \sum_{j=1}^{J} y_{j\ell}$$

*denotes the aggregate amount of good $\ell$ in the economy (coming from endowments and production);*

2. $\omega_i \gg 0$ *for every $i$ (each consumer has a strictly positive endowment of every good);*

3. *preferences are convex*[6]*, continuous, and strictly increasing, where convexity actually used here is: if $x_i, x_i' \in X_i$, $x_i \neq x_i'$, and $x_i \sim_i x_i'$, then for all $\lambda \in (0, 1)$ we have*

$$\lambda x_i + (1 - \lambda) x_i' \succ_i x_i, x_i'.$$

[6] Tillman uses *convex* here but the truth is that they should be *strctly convex*. Nothing less, nothing more.

*Intuition:* mixing two *indifferent* but distinct bundles yields a bundle that is *strictly* preferred.

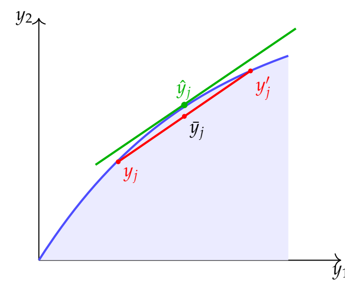
4. *each production set $Y_j$ is compact, convex*[7]*, and satisfies $0 \in Y_j$ for every $j$, where convexity used for production is: if $y_j, y'_j \in Y_j$ with $y_j \neq y'_j$ and $\lambda \in (0,1)$, then*

$$\lambda y_j + (1-\lambda)y'_j \in Y_j \qquad \textbf{and} \qquad \exists \hat{y}_j \in Y_j : \hat{y}_j \gg \lambda y_j + (1-\lambda)y'_j$$

*then the associated excess demand function $z : \Delta \to \mathbb{R}^L$ is well defined and satisfies conditions (1)–(3) above, so there exists at least one Walrasian equilibrium.*

[7] Tillman uses *convex* here but the truth is that they should be *strctly convex*. Nothing less, nothing more.

In words, if two production plans are feasible, then any convex mixture of them is also feasible. Moreover, the technology allows another plan that strictly dominates this mixture, so with strictly positive prices the profit–maximizing plan is unique.




*Economic interpretation of the assumptions.* Assumption (i) puts every consumer in a common "box" $[0, m]^L$ that is large enough to contain any feasible allocation generated by initial endowments and production. This guarantees that demand is always chosen from a compact set that does not depend on prices.

Assumption (ii) requires strictly positive endowments, which guarantees that every consumer has strictly positive wealth at any strictly positive price vector $p \gg 0$ and that each good is present in strictly positive aggregate amount in the economy. Together with strict monotonicity, this rules out degenerate cases in which a consumer would optimally choose a bundle that does not exhaust her budget: for any $p \gg 0$, the solution to the consumer problem satisfies $p \cdot x_i = p \cdot \omega_i$. It also ensures that no good can be "ignored" in equilibrium simply because nobody is initially endowed with it.

**Remark** (Convexity and marginal utilities)**.** Convex, continuous, strictly increasing preferences admit a utility representation $u_i$ that is *quasi-concave*. Quasi-concavity means that all upper contour sets $\{x \in X_i : u_i(x) \geq \bar{u}\}$ are convex. When $u_i$ is differentiable, this can be interpreted in terms of marginal utilities: along any line in the consumption space, marginal utility is "diminishing" so the Hessian of $u_i$ is negative semidefinite. This is the sense in which the convexity assumption above is related to properties of marginal utilities.

Assumption (iii) ensures good behavior of individual demand. Compact, convex budget sets and convex, continuous, strictly increasing preferences imply that each consumer's demand correspondence is nonempty, convex-valued and, by Berge's Maximum Theorem, *upper hemicontinuous* in prices. The extra strict convexity along indifference sets rules out non-unique tangencies: the demand of each consumer is in fact a *single-valued* and continuous function of prices.[8]

[8] Upper hemicontinuity of individual demand correspondences (and of firms' profit-maximizing correspondences) can be obtained from Berge's Maximum Theorem. With the strict convexity ingredients that give uniqueness of the maximizers, upper and lower hemicontinuity collapse into *continuity* of the demand and supply functions. Summing across agents then yields a continuous aggregate excess demand $z(p)$.

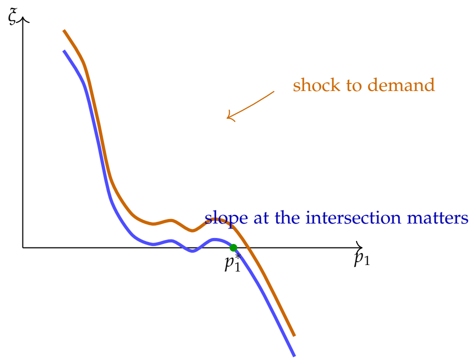
Assumption (iv) plays the analogous role for firms. Compactness of $Y_j$ prevents profits from going off to infinity, so each firm has at least one profit-maximizing production plan at any price. Convexity of $Y_j$ implies that if two plans are feasible, then all mixtures of them are feasible as well. With linear profits $p \cdot y_j$, this delivers a convex maximization problem.

## *Comparative statics and aggregation of supply*

### *Comparative statics*

Consider a parameter $\xi$ that shifts the (aggregate) demand side of the economy. In the $(\xi, p_1)$–plane we can think of the equilibrium price

$p_1^*(\xi)$ as the point where this parameter–dependent demand curve intersects the (fixed) supply curve.




Comparative statics in $(\xi, p_1)$ space. The blue curve gives the locus of $(\xi, p_1)$ pairs that satisfy the equilibrium condition. A positive demand shock shifts this curve upward (orange); the impact on the equilibrium price $p_1^*$ depends on the slope of the curve at the intersection with $\xi = 0$.

A small *shock to demand* corresponds to a small change in $\xi$, which shifts the demand curve. The induced change in the equilibrium price $p_1^*(\xi)$ depends on the *slope of the excess–demand function at the intersection point*: if the curve is steep at the equilibrium, a given horizontal shift in demand produces only a small movement in $p_1^*$, whereas if the curve is relatively flat at that point, the same shift in $\xi$ generates a larger change in the equilibrium price.

### *Aggregation question: aggregation of firms' supply correspondences*

We now turn to the aggregation of firms. Let there be $J$ firms with production sets

$$Y_1, \ldots, Y_J.$$

For a given price vector $p$, the supply correspondence of firm $j$ is

$$S_j(p) = \{y_j \in Y_j \mid p \cdot y_j \geq p \cdot \tilde{y}_j \text{ for all } \tilde{y}_j \in Y_j\}.$$

The aggregate supply correspondence of these $J$ firms at price $p$ is defined as the Minkowski sum

$$S(p) = \sum_{j=1}^{J} S_j(p) = \Big\{ \sum_{j=1}^{J} y_j \,\Big|\, y_j \in S_j(p) \text{ for all } j \Big\}.$$

**Proposition.** *Let $Y_1, \ldots, Y_J$ be the firms' production sets and define the aggregate production set*

$$Y \;=\; Y_1 + Y_2 + \cdots + Y_J \;=\; \Big\{ \sum_{j=1}^{J} y_j \,\Big|\, y_j \in Y_j \textit{ for all } j \Big\}$$

*(the Minkowski sum of the $Y_j$'s). Let $\widehat{S}(p)$ be the supply correspondence of a* single *firm with production set $Y$:*

$$\widehat{S}(p) = \{y \in Y \mid p \cdot y \geq p \cdot \tilde{y} \textit{ for all } \tilde{y} \in Y\}.$$

*Then, for every price vector $p$,*

$$\widehat{S}(p) \;=\; S_1(p) + S_2(p) + \cdots + S_J(p).$$

*Proof.* We need to show the two inclusions

$$\widehat{S}(p) \subseteq \sum_{j=1}^{J} S_j(p) \quad \text{and} \quad \sum_{j=1}^{J} S_j(p) \subseteq \widehat{S}(p).$$

*(i)* $\widehat{S}(p) \subseteq \sum_{j=1}^{J} S_j(p)$. Take $y \in \widehat{S}(p)$. By definition of $Y = Y_1 + \cdots + Y_J$, there exist vectors $y_j \in Y_j$ such that

$$y = \sum_{j=1}^{J} y_j.$$

We claim that in fact $y_j \in S_j(p)$ for every $j$.

Suppose not. Then for some firm $j$ there exists $y_j' \in Y_j$ with

$$p \cdot y_j' > p \cdot y_j.$$

Consider the alternative aggregate production vector

$$y' = \sum_{k \neq j} y_k + y_j' \in Y.$$

Its profit is

$$p \cdot y' = p \cdot \Big( \sum_{k \neq j} y_k + y_j' \Big) > p \cdot \Big( \sum_{k \neq j} y_k + y_j \Big) = p \cdot y,$$

which contradicts $y \in \widehat{S}(p)$, since $y$ was supposed to maximize profits over $Y$. Hence our assumption was false, and we must have $y_j \in S_j(p)$ for all $j$. Therefore $y = \sum_{j=1}^{J} y_j \in \sum_{j=1}^{J} S_j(p)$.

*(ii)* $\sum_{j=1}^{J} S_j(p) \subseteq \widehat{S}(p)$. Now take vectors $y_j \in S_j(p)$ for all $j$, and let

$$y = \sum_{j=1}^{J} y_j.$$

By construction $y \in Y$. We show that $y \in \widehat{S}(p)$.

Suppose not. Then $y \notin \widehat{S}(p)$, so there exists some $y' \in Y$ with

$$p \cdot y' > p \cdot y.$$

Because $y' \in Y = Y_1 + \cdots + Y_J$, there are vectors $y_j' \in Y_j$ such that

$$y' = \sum_{j=1}^{J} y_j'.$$

Therefore

$$p \cdot \sum_{j=1}^{J} y'_j = p \cdot y' > p \cdot y = p \cdot \sum_{j=1}^{J} y_j.$$

This strict inequality of sums implies that for at least one firm $j$ we must have

$$p \cdot y'_j > p \cdot y_j,$$

contradicting $y_j \in S_j(p)$ (since $S_j(p)$ is the set of maximizers of $p \cdot y_j$ over $Y_j$). Thus no such $y'$ can exist, and $y$ must belong to $\widehat{S}(p)$.

Combining (i) and (ii) we conclude that $\widehat{S}(p) = S_1(p) + \cdots + S_J(p)$ for every price vector $p$. □

# *Class 11: Aggregate demand functions*

In the previous class we started from the primitives of the economy and derived an excess demand function $z : \Delta \to \mathbb{R}^L$ that is continuous, homogeneous of degree zero and satisfies Walras' Law. We now move one step *forward*: instead of asking which primitives deliver a well–behaved excess demand function, we ask what properties of *individual* demand carry over to the *aggregate* demand of an economy.

## *Monotonicity and aggregation*

Recall the "law of demand" type condition that arises from the Slutsky matrix.

**Definition** (Monotonicity in the Slutsky sense)**.** A function $f : \mathbb{R}^L_{++} \to \mathbb{R}^L$ is *monotone* (in the Slutsky sense) if for every pair of price vectors $p, p' \gg 0$ we have

$$(p' - p) \cdot \big(f(p') - f(p)\big) \;\leq 0.$$

For Hicksian (compensated) demand $h(p, u)$ this inequality always holds: a change in prices that raises the cost of the consumption bundle cannot increase the compensated quantity demanded.

*Law of demand:* a price increase, holding utility fixed, cannot raise the value of demand at the new prices. Equivalently, the Slutsky matrix is negative semidefinite.

By contrast, Marshallian (uncompensated) demand $x(p, m)$ need not be monotone: income effects can overturn the pure substitution effect and violate the law of demand. Hence *individual demand correspondences need not be monotone*; only compensated ones satisfy the Slutsky monotonicity property.

A simple but important observation is that monotonicity is preserved by aggregation.

**Proposition** (Monotonicity survives summation)**.** *Let $f, g : \mathbb{R}^L_{++} \to \mathbb{R}^L$ be two monotone functions in the sense above. Then their sum $f + g$ is also monotone. More generally, the sum of any finite collection of monotone functions is monotone.*

*Proof.* For any $p, p' \gg 0$,

$$\begin{aligned}(p'-p)\cdot\big[(f+g)(p')-(f+g)(p)\big] = \\ &= (p'-p)\cdot\big(f(p')-f(p)\big)+(p'-p)\cdot\big(g(p')-g(p)\big) \\ &\leq 0+0=0\end{aligned}$$

since each term is nonpositive by monotonicity of $f$ and $g$. The extension to finitely many functions is immediate by induction. □

**Remark.** If each consumer's (Hicksian) compensated demand $h_i(p, u_i)$ is monotone, then the aggregate compensated demand

$$h(p, u_1, \ldots, u_I) = \sum_{i=1}^{I} h_i(p, u_i)$$

is also monotone. Thus the law of demand for compensated demand survives aggregation. What will turn out to be much more delicate is which properties of *Marshallian* demand survive aggregation.

## *Income aggregation and the representative consumer*

We now ask when the aggregate Marshallian demand of many consumers can be represented as if it were the demand of a single "representative consumer" whose only characteristic is *aggregate income*.

For consumer $i$ let

$$D_i : \mathbb{R}^L_{++} \times \mathbb{R}_+ \to \mathbb{R}^L_+, \qquad (p, y_i) \longmapsto D_i(p, y_i)$$

denote her Marshallian demand (for given prices $p$ and income $y_i$). For a profile of incomes $(y_1, \ldots, y_I)$ the aggregate demand of the $I$ consumers is

$$\sum_{i=1}^{I} D_i(p, y_i).$$

**Definition** (Income aggregation)**.** We say that *income aggregation* holds if there exists a function

$$D : \mathbb{R}^L_{++} \times \mathbb{R}_+ \to \mathbb{R}^L_+$$

such that, for every price vector $p$ and every income profile $(y_1, \ldots, y_I)$,

$$\sum_{i=1}^{I} D_i(p, y_i) \;=\; D\Big(p, \sum_{i=1}^{I} y_i\Big).$$

In words, the aggregate demand depends only on prices and *aggregate* income $Y = \sum_i y_i$, not on how this income is distributed across consumers.

*Idea:* if income aggregation holds, the entire economy can be summarized by a single demand function $D(p, Y)$, as if there were one representative consumer with income $Y$.

The following proposition characterizes when such a representative consumer exists.

**Proposition** (Characterization of income aggregation)**.** *There exists a function $D(p, Y)$ satisfying income aggregation if and only if the following two conditions hold:*

*(i)* ***Symmetry across consumers:*** *For all $i, j$,*

$$D_i(p, y) = D_j(p, y) \quad \text{for all } (p, y) \in \mathbb{R}^L_{++} \times \mathbb{R}_+.$$

*That is, at any given price vector and income level, consumers with the same income have the same demand bundle.*

*(ii)* ***Linear Engel curves:*** *There exists a function $\alpha : \mathbb{R}^L_{++} \to \mathbb{R}^L_+$ such that for every consumer $i$ and every $(p, y_i)$,*

$$D_i(p, y_i) = y_i\, \alpha(p),$$

*in particular $D_i(p, 0) = 0$ for all $i$ and $p$.*

**Remark.** Condition (ii) means that each component of demand is *proportional* to income at any fixed price vector $p$. The vector $\alpha(p)$ captures the shares of income spent on each good. Thus all consumers share the same Engel curve (the same income expansion path), which is a ray from the origin. This is equivalent to assuming identical homothetic preferences across consumers. Under (i)–(ii) we have

$$\sum_{i=1}^{I} D_i(p, y_i) = \sum_{i=1}^{I} y_i\, \alpha(p) = \Big(\sum_{i=1}^{I} y_i\Big)\alpha(p),$$

so income aggregation holds with $D(p, Y) = Y\alpha(p)$.

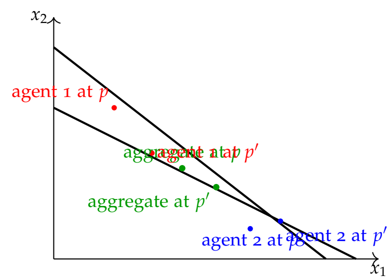
## *Aggregation and the WARP*

Even when a representative consumer exists in the sense of income aggregation, the aggregate demand need not satisfy the Weak Axiom of Revealed Preference (WARP).

Consider an economy with only two goods and $I = 2$ consumers. Let the income of consumer $i$ be a fixed share $\beta_i > 0$ of total income $Y$, with $\beta_1 + \beta_2 = 1$, so that $y_i = \beta_i Y$.

Assume that both consumers have identical, well–behaved preferences that generate Marshallian demands $D_i(p, y_i)$ satisfying WARP. Under the conditions for income aggregation above, aggregate demand is proportional to $Y$ and can be represented by a single function $D(p, Y)$.

The figure illustrates that even though each agent's demand satisfies WARP, the *aggregate* demand can violate it: there exist price vectors $p$




Aggregation and the WARP. Each individual satisfies WARP, but the aggregate choices (green points) can violate it. At both price vectors $p$ and $p'$ each green bundle is affordable, yet the aggregate demand switches from one to the other, contradicting WARP.

and $p'$ and aggregate bundles $x(p)$ and $x(p')$ such that each bundle is affordable at both prices and yet the aggregate chooses $x(p)$ at $p$ and $x(p')$ at $p'$, contradicting WARP. Thus revealed–preference consistency is *not* a property that survives aggregation, even under strong symmetry and homotheticity assumptions.

## *The Sonnenschein–Mantel–Debreu theorem*

We return to the aggregate *excess demand* function of an exchange economy. Let $\xi(p)$ denote the excess demand function associated with some economy.

From last class we know that under the standard assumptions on preferences, endowments and technologies, $\xi$ satisfies three key properties:

(i) **Continuity:** $\xi(p)$ is continuous in $p$.

(ii) **Homogeneity of degree zero:** $\xi(\lambda p) = \xi(p)$ for all $\lambda > 0$.

(iii) **Walras' Law:** $p \cdot \xi(p) \equiv 0$ for all $p \gg 0$.

A remarkable result due to Sonnenschein, Mantel and Debreu says that, apart from (i)–(iii), essentially *no further restrictions* on $\xi(p)$ are implied by standard assumptions at the individual level.

*Question:* are there other economically meaningful restrictions on individual behavior that necessarily survive aggregation at the level of $\xi(p)$? We have just seen that even WARP does not.

**Theorem** (Sonnenschein–Mantel–Debreu). *Let $\xi : \mathbb{R}^L_{++} \to \mathbb{R}^L$ be any function that is continuous, homogeneous of degree zero and satisfies Walras' Law, $p \cdot \xi(p) \equiv 0$. Then there exists an exchange economy (with a suitable number of consumers and endowments) whose aggregate excess demand function is precisely $\xi(p)$. In that economy, no additional property of individual demand—such as WARP, Slutsky symmetry, or monotonicity of Marshallian demand—survives aggregation.*

# *Class 12: Uniqueness of equilibrium*

We now ask whether the Walrasian equilibrium we have shown to exist is *unique*. More precisely, if $\xi(p)$ is the aggregate excess demand function, we ask whether the equation

$$\xi(p) = 0, \qquad p \in \Delta,$$

has a unique solution. There are two logically distinct questions:

1. uniqueness of *prices* (up to normalization), and
2. uniqueness of *quantities* (equilibrium allocation).

In what follows we first obtain a result on uniqueness of aggregate quantities, and then study uniqueness of prices.

## *Uniqueness of quantities with a representative consumer*

**Proposition.** *Suppose the First Welfare Theorem holds (with its usual assumptions). Suppose there exists a representative consumer with strictly convex preferences, and suppose that the aggregate production set*

$$\sum_{j=1}^{J} Y_j \;=\; \Big\{ \sum_{j=1}^{J} y_j : y_j \in Y_j \Big\}$$

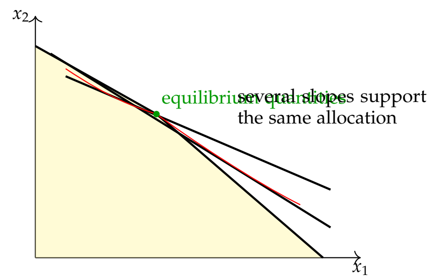
*is convex. Then the Walrasian equilibrium aggregate quantities are unique.*

**Remark.** The feasible set of aggregate net outputs is

$$\omega + \sum_{j=1}^{J} Y_j,$$

which is convex by assumption. Strict convexity of the representative consumer's preferences implies that the social optimum is a unique point of this convex set. By the First Welfare Theorem, any Walrasian equilibrium allocation coincides with such a social optimum, so the aggregate equilibrium quantities are unique. However, the supporting *prices* for this unique allocation need not be unique: if the frontier of

$\omega + \sum_j Y_j$ has a kink at the equilibrium allocation, there can be many supporting hyperplanes (and thus many relative price vectors) passing through that point.




Unique equilibrium quantities but possibly multiple supporting price vectors when the aggregate technology set has a kink.

## *Gross substitutes and uniqueness of prices*

To obtain uniqueness of *prices*, we need a stronger restriction on the shape of the excess demand function. We will assume that all goods are *gross substitutes*.

Because of homogeneity of degree zero, we can think of the excess demand as a function defined on the whole positive orthant, not just on the simplex:

$$\xi : \mathbb{R}^L_{++} \to \mathbb{R}^L,$$

and then normalize prices back to $\Delta$ when needed. Working on $\mathbb{R}^L_{++}$ allows us to vary one price at a time while holding the others fixed and to speak of partial derivatives with respect to individual prices.

**Gross substitutes assumption.** We assume that $\xi$ is continuously differentiable and satisfies

$$\frac{\partial \xi_\ell(p)}{\partial p_\ell} < 0 \quad \text{and} \quad \frac{\partial \xi_\ell(p)}{\partial p_k} > 0 \quad \text{for all } \ell \neq k.$$

An increase in the price of good $\ell$ reduces its own excess demand (own-price effect negative), while an increase in the price of any other good $k$ raises the excess demand for good $\ell$ (goods are gross substitutes in excess-demand sense).

**Proposition.** *Suppose $\xi : \mathbb{R}^L_{++} \to \mathbb{R}^L$ is homogeneous of degree zero, continuously differentiable, and satisfies the gross–substitutes sign conditions above. Then there is at most one $p^* \in \Delta$ with $p^* \gg 0$ such that*

$$\xi(p^*) = 0.$$

*In other words, the Walrasian equilibrium price vector (up to normalization) is unique.*

*Proof.* Suppose, towards a contradiction, that there exist two equilibrium price vectors $p', p'' \in \Delta$ with $p', p'' \gg 0$ such that

$$\xi(p') = 0, \qquad \xi(p'') = 0,$$

and $p''$ is not a positive scalar multiple of $p'$ (so $p'' \neq \lambda p'$ for all $\lambda > 0$).

Define

$$\lambda \equiv \max_{\ell=1,\dots,L} \frac{p'_\ell}{p''_\ell},$$

and let $\hat{\ell}$ be an index at which this maximum is attained. Then

$$p'_{\hat{\ell}} = \lambda p''_{\hat{\ell}}, \qquad p'_\ell \leq \lambda p''_\ell \quad \text{for all } \ell,$$

and there is at least one $k$ such that $p'_k < \lambda p''_k$ (otherwise $p' = \lambda p''$).

Consider the price vector $q \equiv \lambda p''$. By homogeneity of degree zero,

$$\xi(q) = \xi(\lambda p'') = \xi(p'') = 0.$$

Notice that

$$q_{\hat{\ell}} = p'_{\hat{\ell}}, \qquad q_k \geq p'_k \ \text{ for all } k,$$

and for at least one $k \neq \hat{\ell}$ we have $q_k > p'_k$.

Now look at the excess demand for good $\hat{\ell}$. When moving from $p'$ to $q$, the own price of good $\hat{\ell}$ is unchanged, but the prices of at least one other good $k \neq \hat{\ell}$ increase. Since

$$\frac{\partial \xi_{\hat{\ell}}}{\partial p_k} > 0 \quad \text{for all } k \neq \hat{\ell},$$

raising some of the other prices strictly increases the excess demand for good $\hat{\ell}$. Hence

$$\xi_{\hat{\ell}}(q) > \xi_{\hat{\ell}}(p').$$

On the other hand, $\xi(q) = 0$ implies $\xi_{\hat{\ell}}(q) = 0$, so the previous inequality yields

$$0 = \xi_{\hat{\ell}}(q) > \xi_{\hat{\ell}}(p'),$$

and therefore $\xi_{\hat{\ell}}(p') < 0$. This contradicts the assumption that $p'$ is an equilibrium price vector, which would require $\xi(p') = 0$.

We conclude that there cannot exist two distinct price vectors in $\Delta$ that solve $\xi(p) = 0$. Thus the Walrasian equilibrium price vector is unique (up to normalization). □

## *Convexity of the set of equilibrium prices*

We now study an alternative route to uniqueness-type results, based on revealed preference restrictions on excess demand and constant returns to scale on the production side.

Assume that the aggregate excess demand function $z : \Delta \to \mathbb{R}^L$ is single–valued and that the firms' production sets satisfy constant returns to scale (CRS).

**Proposition.** *If the following conditions hold:*

(i) *For every consumer $i$ and every price vector $p \in \Delta$ there is a unique bundle $x_i(p)$ that maximizes $\succeq_i$ subject to the budget constraint $p \cdot x_i \leq p \cdot \omega_i$, and*

$$z(p) = \sum_{i=1}^{I} (x_i(p) - \omega_i);$$

(ii) *$z$ satisfies Walras' Law:*

$$p \cdot z(p) = 0 \qquad \forall p \in \Delta;$$

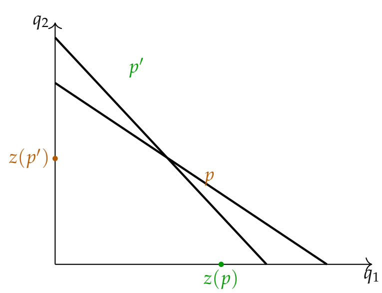
(iii) *$z$ satisfies WARP, in the sense that for any $p, p' \in \Delta$,*

$$p \cdot z(p') \leq 0 \quad \text{and} \quad p' \cdot z(p) \leq 0 \quad \Rightarrow \quad z(p) = z(p');$$

(iv) *For every firm $j$ the production set $Y_j$ has CRS:*

$$y_j \in Y_j \text{ and } \lambda \geq 0 \quad \Rightarrow \quad \lambda y_j \in Y_j,$$

*then the set of Walrasian equilibrium price vectors is convex.*




Under WARP for aggregate excess demand and CRS on the production side, the set of Walrasian equilibrium prices is convex.

## *Regular equilibria and the Implicit Function Theorem*

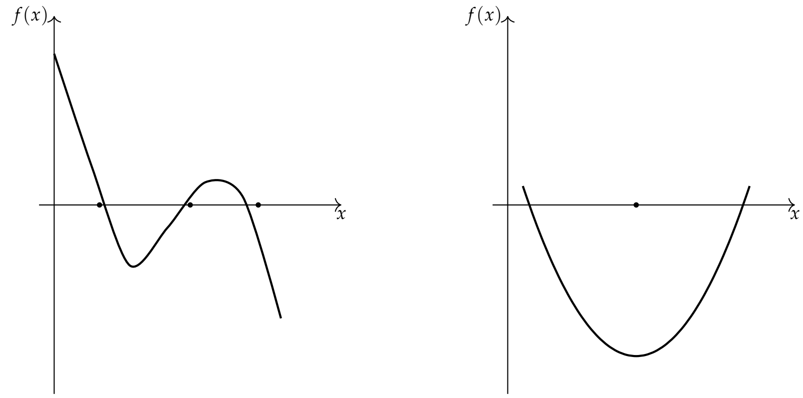
To understand when equilibrium prices are locally unique, we recall the Implicit Function Theorem.

### *Implicit Function Theorem (intuition)*

Consider an equation

$$f(x, a) = 0,$$

where $x \in \mathbb{R}^n$ is the unknown and $a \in \mathbb{R}^k$ is a parameter vector. Roughly speaking, if the Jacobian of $f$ with respect to $x$ is invertible at a solution $(x^*, a^*)$, then there is a *locally unique* solution $x(a)$ in a neighborhood of $a^*$; invertibility of this Jacobian is both necessary and sufficient for such local uniqueness.




Left: a function with several zeros. Right: a function with an isolated zero where the derivative is nonzero; the Jacobian is invertible and the solution is locally unique.

*Reduced excess demand and regular equilibria*

For the Walrasian equilibrium problem, we would like to apply the Implicit Function Theorem to the system $\xi(p) = 0$. However, the Jacobian of $\xi$ is never invertible because Walras' Law implies that the rows are linearly dependent. To restore invertibility we normalize one price and work in dimension $L - 1$.

Let us normalize the last good as numéraire, $p_L \equiv 1$, and define the *reduced excess demand* map

$$\eta : \mathbb{R}^{L-1}_{++} \to \mathbb{R}^{L-1}, \qquad \eta(p_{-L}) \equiv \big(\xi_1(\tilde{p}), \ldots, \xi_{L-1}(\tilde{p})\big),$$

where $\tilde{p} = (p_{-L}, 1)$ is the corresponding $L$–dimensional price vector. Equilibrium in the original economy is equivalent to solving $\eta(p_{-L}) = 0$ in $\mathbb{R}^{L-1}_{++}$.

**Proposition.** *Suppose $\eta(p^*) = 0$ for some $p^* \in \mathbb{R}^{L-1}_{++}$ and the Jacobian $D\eta(p^*)$ is invertible. Then there exists $\varepsilon > 0$ such that*

$$p \neq p^*, \quad \|p - p^*\| \leq \varepsilon \quad \Rightarrow \quad \eta(p) \neq 0.$$

*In particular, $p^*$ is an isolated solution of $\eta(p) = 0$. The corresponding price vector $\tilde{p}^* = (p^*, 1)$ is called a* regular equilibrium*.*

**Proposition.** *For "almost all" (or* generic*) economies, the set of Walrasian equilibrium price vectors in $\Delta$ is finite and consists of an odd number of points $(1, 3, 5, \ldots)$. Moreover, all these equilibrium price vectors are regular.*

# Class 13: Local Uniqueness

Global uniqueness is very restrictive. We now study *local* uniqueness of Walrasian equilibrium prices by again considering the reduced excess demand map under a price normalization.

Let

$$\eta : \mathbb{R}^{L-1}_{++} \to \mathbb{R}^{L-1}$$

be continuously differentiable, and suppose we normalize the last good as numéraire so that $p = (p_{-L}, 1)$. A Walrasian equilibrium in reduced form solves

$$\eta(p) = 0.$$

**Proposition.** *Suppose $\eta$ is continuously differentiable and let $p^*$ satisfy $\eta(p^*) = 0$. If the Jacobian $D_p\eta(p^*)$ is invertible, then $p^*$ is locally unique.*

The invertibility of the Jacobian allows us to apply the Implicit Function Theorem. Hence, in a neighborhood of $p^*$, there is no other price vector solving the reduced equilibrium conditions.

## *Comparative statics of equilibrium prices*

We now introduce a parameter $q$ and consider the parameterized reduced excess demand,

$$\eta(p, q).$$

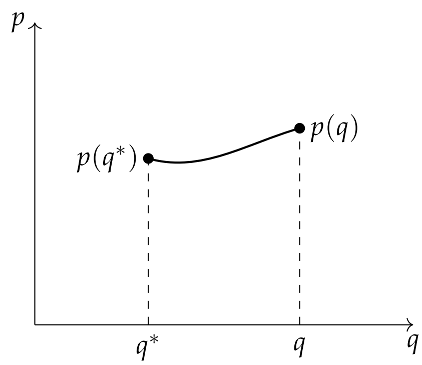
A change in $q$ affects the equilibrium through the equation

$$\eta(p(q), q) = 0.$$

Suppose the equilibrium $p(q^*)$ is locally unique. Then $p(q)$ is our object of interest, particularly the derivative

$$\frac{\partial p}{\partial q}.$$

Assume for simplicity that $D_p\eta(p^*, q^*)$ is invertible. Local uniqueness ensures we can solve for $p(q)$ smoothly in a neighborhood of $q^*$.




Local function $p(q)$ around $q^*$. Local uniqueness rules out jumps.

Local uniqueness ensures a local function $p(q)$ around $q^*$. We do *not* study jumps in equilibrium prices here; the IFT provides a differentiable selection.

*Monotonicity*

Consider the monotonicity condition:

$$(p(q) - p(\hat{q})) \cdot (q - \hat{q}) \geq 0.$$

Via the IFT, this corresponds to checking the sign of the directional derivative:

$$(D_q p(q) \cdot \Delta q) \cdot (D_q \eta(p(q), q) \cdot \Delta q) \geq 0 \qquad \forall\, \Delta q.$$

If $D_p \eta(p^*, q^*)$ satisfies a monotonicity property, then monotonicity of $p(q)$ follows.

This holds, for example, in an exchange economy with a representative consumer (without production). The intuition is that excess demand increasing in $q$ induces prices to increase, though the exact mechanism is nontrivial.

## *Gross substitutes and monotonicity*

Suppose now that all goods are gross substitutes. Then for all $\ell \neq k$:

$$\frac{\partial \eta_\ell(p,q)}{\partial p_k} > 0, \qquad \frac{\partial \eta_\ell(p,q)}{\partial p_\ell} < 0.$$

This is a more restrictive assumption, but under it the Jacobian $D_p \eta(p, q)$ is a P–matrix, hence invertible and sign–preserving.

Thus, if

$$D_q \eta(p^*, q^*) \cdot \Delta q \gg 0,$$

then

$$D_q p(q^*) \cdot \Delta q \gg 0.$$

So under gross substitutes, a monotone increase in the parameter $q$ induces a monotone increase in equilibrium prices.

## *Time and Uncertainty in General Equilibrium*

We now extend the general–equilibrium framework to environments in which both time and uncertainty play a central role. This allows us to analyze economies in which markets open before uncertainty is resolved, and consumption takes place only after the relevant state of the world becomes known. A central distinction in this setting is between *complete* and *incomplete* markets.

### *A simple intertemporal–uncertainty model*

Consider an economy that unfolds over two periods, $t = 0$ and $t = 1$. Uncertainty is resolved in period $t = 1$, and we denote by

$$s \in S = \{1, 2, \dots, S\}$$

the realized state of the world at that time. All trading of financial or contingent claims occurs in the initial period $t = 0$, while physical consumption takes place only in period $t = 1$. This structure captures the idea that individuals make ex–ante choices under uncertainty and then realize consumption once the uncertainty has been resolved.

Consumers may differ in their attitudes toward risk and therefore in how they evaluate state– contingent consumption plans.

There are $I$ consumers, indexed by $i = 1, \dots, I$. Each consumer's feasible consumption plan specifies consumption in period 1 in each possible state $s \in S$. We therefore represent consumer $i$'s consumption set as

$$\mathbb{R}_+^{2\times S},$$

where the two components in each state correspond to two physical goods.

Preferences are defined over these state–contingent consumption bundles. Let $\succeq_i$ denote consumer $i$'s preference relation on $\mathbb{R}_+^{2\times S}$. We impose two standard assumptions that facilitate the analysis.

**Assumption 1.** Preferences $\succeq_i$ are strictly increasing: consumers always prefer more of any good in any state.

**Assumption 2.** For every state $s \in S$, there exists a well–defined conditional preference relation $\succeq_{i,s}$ over $\mathbb{R}_+^2$. This condition represents a form of separability across states: the consumer can evaluate state–specific consumption bundles and aggregate them into overall preferences.

These assumptions are minimal yet sufficient to embed uncertainty into the general–equilibrium framework in a way that preserves tractability while allowing for rich heterogeneity in risk attitudes and in responses to uncertainty.

# *Class 14: Time and Uncertainty*

### *Basic Structure of the Environment*

We consider a two–period economy, $t = 0, 1$, in which uncertainty is resolved in period 1. There are $S$ possible states of the world, indexed by

$$s \in \{1, \dots, S\}.$$

Consumers trade at $t = 0$ over state–contingent commodities

$$x_i \in \mathbb{R}_+^{2 \times S},$$

and hold endowments

$$\omega_i \in \mathbb{R}_+^L.$$

At $t = 1$ one of the states $s$ materializes. Consumption takes place after the realization.

There is time consistency across states, and preferences are strictly increasing. Our goal is to examine how consumers trade ex–ante under different uncertainty scenarios.

### *Arrow–Debreu Interpretation*

The key reinterpretation is the following:

> A *commodity* is defined as: "one unit of good $\ell$ delivered at time $t$ *if and only if* the realized state is $s_t$."

Thus, the period–0 market trades the full vector of $L \times S$ Arrow–Debreu commodities. If this market is complete, then the standard welfare theorems apply exactly as in the timeless general–equilibrium model.

*No Trade Result under Identical Conditional Preferences* If consumers have identical preferences *in each state* $s$, then after the realization of uncertainty there is no scope for further trades: all relevant reallocations were already anticipated and executed at $t = 0$.

This is the canonical Arrow–Debreu result: ex–ante markets implement all redistributions.

What if preferences are convex but I only learn state–dependent preferences at $t = 1$?

### *The Radner Model*

We now reinterpret the two–period economy with uncertainty using the Radner framework. The key distinction relative to Arrow–Debreu is that markets open sequentially:

1. At period 0, consumers trade a set of $K$ financial assets at prices $q \in \mathbb{R}^K$.

2. At period 1, after the state $s$ is realized, spot markets for goods open with state–contingent prices $p_s \in \mathbb{R}^L_{++}$.

Each asset $k$ has a payoff vector across states described by the return matrix

$$R = \begin{pmatrix} r_{11} & \cdots & r_{1S} \\ \vdots & \ddots & \vdots \\ r_{K1} & \cdots & r_{KS} \end{pmatrix}, \qquad r_{ks} \text{ units of the numéraire delivered in state } s.$$

Thus assets are linear combinations of Arrow–Debreu securities, but only $K$ such combinations are available.

### *Timing and Individual Optimization*

*Period* 0. Consumer $i$ chooses a portfolio $z_i \in \mathbb{R}^K$ satisfying

$$q \cdot z_i \leq 0.$$

*Period* 1. Given realized state $s$, consumer $i$ chooses $x_{is} \in \mathbb{R}^L_+$ subject to

$$p_s \cdot x_{is} \;\leq\; p_s \cdot \omega_{is} + p_{s1} \sum_{k=1}^{K} z_{ik} r_{sk}.$$

This incorporates all intertemporal trades into state–$s$ income expressed in units of the numéraire.

El número de activos determina el subespacio de planes de consumo que puedo implementar. Solo puedo alcanzar combinaciones lineales de las columnas de $R$.

### *Radner Equilibrium*

A *Radner equilibrium* consists of:

- asset prices $q \in \mathbb{R}^K$,
- state–contingent spot prices $p_s \in \mathbb{R}^L_{++}$ for each state $s$,
- portfolios $(z_i)_{i=1}^I$,
- and consumption allocations $(x_{is})_{i=1}^I$,

such that:

1. Given $(q, p_s)$, each consumer maximizes expected utility subject to the budget constraints at $t = 0$ and $t = 1$.
2. Spot markets clear in each state:

$$\sum_{i=1}^{I} x_{is} = \sum_{i=1}^{I} \omega_{is}, \qquad \forall s.$$

3. Asset markets clear at $t = 0$:

$$\sum_{i=1}^{I} z_i = 0.$$

This defines competitive behavior when uncertainty is present and claims markets may be incomplete.

### *Relation to Arrow–Debreu*

- If $\text{rank}(R) = S$ (complete markets), Radner equilibrium allocations coincide with Arrow–Debreu equilibrium allocations.
- If $\text{rank}(R) < S$, markets are incomplete and Radner equilibria need not be Pareto efficient; the First Welfare Theorem may fail.

### *Consumers' Budget Constraints*

At period 0, each consumer chooses a portfolio

$$z_i = (z_{i1}, \ldots, z_{iK}) \in \mathbb{R}^K,$$

subject to the financial budget constraint

$$q \cdot z_i \leq 0.$$

Idea: el número de activos determina qué combinaciones de consumo puedo elegir en cada estado. ¿Es una combinación lineal de los $K$ activos?

*Period 1 Budget Constraint in State s* Once the state $s$ is realized, spot markets for goods open. Consumer $i$ must satisfy:

$$p_s \cdot x_{is} \ \leq \ p_s \cdot \omega_{is} \ + \ p_{s1} \sum_{k=1}^{K} z_{ik} \, r_{sk}.$$

Thus all trades arranged in period 0 are fully incorporated into period–1 income, expressed in units of the numéraire good.

If the number of assets is insufficient relative to the number of states, i.e.

$$K < S,$$

then the *First Welfare Theorem does not hold* in general. The key reason is that markets fail to span all contingencies, so competitive allocations need not be Pareto efficient.

## *Complete vs. Incomplete Markets*

*Complete Markets* Markets are complete whenever the return matrix $R$ has full row rank:

$$\text{rank}(R) = S.$$

In this case:

- the First Welfare Theorem holds, and
- equilibria of the Radner model coincide with equilibria of the Arrow–Debreu model (except for consumption at $t = 0$, which is absent in the Radner formulation).

*Incomplete Markets* If instead

$$\text{rank}(R) < S,$$

then markets are incomplete: assets fail to span all payoff vectors. Radner equilibria may no longer be Pareto efficient, and welfare theorems do not apply.